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THE MAGELLANIC CLOUDS. XIV. THE BAR OF THE LARGE CLOUD

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Communicated January 31, 1955

Of the thousand brightest galaxies, only 3 or 4 per cent are irregular in form, but, of the thousand *nearest* galaxies, the majority of which are too faint to enter the listing of bright galaxies, the percentage of irregular galaxies is much higher, amounting to 25 per cent in the so-called "local family." The irregularities range from those of the chaotic type of galaxy like IC 1613 to one-armed spirals such as NGC 4038–4039 and discoids without central nuclei like NGC 55. Intermediate in its disorganization is the Large Magellanic Cloud, which, dominated by a long stellar axis or bar rather than by a central nucleus, can properly be classed as a quasi-barred spiral.[1] But, unlike the bars in most of the brighter spirals, there is at the center of the bar in the Cloud no bright nucleus or appreciable widening.

General Survey.—The bar of the Large Cloud lies half a degree south of the geometrical center of the Cloud. It covers approximately two and a half square degrees, its angular length being three and a half degrees and the average width one-fifth as much. The linear length and breadth are 10,500 and 2,100 light years, respectively, if the distance is taken as 170,000 light years.[2]

On both blue- and red-sensitive plates the distribution of stars along the bar appears at first glance to be fairly uniform, but close examination shows a loose scattering of giants at the west end of the bar, and all along the bar a large number of clusters of supergiant stars, a few patches of bright nebulosity, and many of obscuration, which are mostly small. The largest dark patches (one extends 750 light years) are at the east end of the bar, south of the 30 Doradus gaseous nebulosity, and may be extensions of that dust and gas complex; otherwise, they show no particular dependence on position along the bar or affiliation with bright nebulosities and clusters.

The characteristics of the star clusters in the bar area can be summarized as follows: 4 probably globular;[3] 9 circular but probably not globular; 19 open, without conspicuous nebulosity; 24 open, with bright nebulosity.

More than half the approximately 150 nonvariable stars which have apparent magnitudes between 13 and 14 (scattered along the bar) are superposed galactic stars. Those that actually belong to the Cloud are supergiants, with absolute magnitudes in excess of -5; they are in part associated with the open clusters. Some are the irregular red variables discussed in an earlier paper,[4] and one is the well-known P Cygni–type variable S Doradus, with a luminosity approximately a million

times that of the sun. It is involved in a striking S-shaped bright nebulosity that is over five hundred light years in diameter. The giant cepheid variables, which are abundantly found all along the bar, are discussed below.

The *Henry Draper Extension*[5] gives the spectra of 144 stars along this axis of the Cloud as follows: Class O 2, B 17, A 7, F 21, G 38, K 54, M 3, and Peculiar 2. They are all brighter than 13.5 mag. From the Van Rhijn tables we find that over the area of the bar approximately 80 stars per square degree to magnitude 13.5 would be expected in the foreground. Therefore, probably only the O, B, and perhaps some of the fainter K's are Cloud members. Other preliminary studies of spectral class and color have indicated that the spectral types of these supergiants range from the extreme blue to the extreme red.[6]

The Noncepheid Variables. The stars listed in Table 1 have been identified as eclipsing, long-period, RR Lyrae, or irregular variables. (For positions of variables consult the index *in Harvard Bull.* 919, 1949.) Elsewhere we shall publish a list of stars that were once suspected of variation and are now proved not appreciably variable and a tabulation of data on variables of the bar that have been measured but for which no certain type or period is obtained. Some are near or involved in dark nebulosity; some have close companions; the majority have ranges of only about half a magnitude. A number of variables, of course, remain undiscovered. Presumably some RR Lyrae stars should be found with magnitudes fainter than 18.5. Numerous photographs with the 60-inch Rockefeller reflector,

TABLE 1

HV No.	Magnitudes Max.	Min. 1	Min. 2	Period (Days)
		Eclipsing		
910	15.35	16.1	15.8	1.331
936	14.55	15.85	14.68	10.032
981	14.6	15.8	...	8.486*
2329	15.3	16.2	15.5	4.176
2348	13.85	14.85	14.45	1.990
2436	15.75	16.60	15.90	3.011
2597	15.8	16.7	15.95	0.645
2608	15.4	16.4	15.7	2.917
5581	15.5	16.5	15.9	5.916
5622	14.5	15.1	....	....
5669	15.9	16.8	....	4.591
5963	14.75	15.6	15.35	3.257
11981	16.3	17.0	...	1.511
11999	14.7	16.7	...	2.008
12228	16.2	16.9	...	....
12324	16.3	17.0	16.5	2.656
		Long Period		
2493	15.8	17.5		552.5
5680	16.3	[17.0		303±
5810	15.5	[17.0		359±
12048	14.8	[17.4		540
		RR Lyrae		
2323	14.9	16.0		0.6882
2397	15.4	16.2		....
2492	13.2	14.6		0.5531
5557	13.0	13.8		....
5602	12.3	13.0		....
5651	14.5	15.3		....
5733	15.0	16.0		....

* Period may be 4.243 days.

HV No.	Magnitudes Max.	Min.
	Irregular	
915	14.4	15.8
970	15.4	16.1
1011	15.0	16.5
2306	14.6	15.5
2377	14.2	15.3
2475	14.2	15.2
2479	14.5	15.7
2554	16.0	16.8
2576	16.0	[17.0
2578	15.4	[17.3
2670	14.3	14.9
5582	12.5	13.5
5625	14.4	15.6
5700	16.0	16.8
5720	15.2	16.0
5717	13.8	15.6
5691	14.7	16.0
5763	14.5	15.4
5769	16.3	[17.0
5772	14.7	15.6
5781	15.0	15.5
5868	14.5	15.3
11984	15.2	17.2
12012	14.8	15.4
12013	14.6	15.7
12014	16.2	16.8
12017	14.4	15.7
12343	14.5	15.7
	Nova	
12336	13.3	...

however, have failed to add appreciably to the number of variables discovered on the Bruce plates, and a preliminary examination of the limited number of ADH photographs also has added little. To magnitude 16.5 the material is probably 80 per cent complete. Whatever the lack in fulness of discovery may be, it should not appreciably affect the intercomparison of the magnitudes and periods of the classical cepheids along the bar or endanger the deductions derived therefrom.

In the area of the bar we have identified seven RR Lyrae variables with median magnitudes brighter than 15.8, which we have naturally assumed to be superposed galactic stars. Assuming the median absolute magnitude near zero, we find they are all less than one-third the distance to the Cloud.

The New Cepheid Variables.—In the course of the present study the new variables listed in Table 2 have been discovered and are now for the first time announced.

TABLE 2

HV No.	Co-ordinates X	Co-ordinates Y	Log of Period	Median Magnitude	HV No.	Co-ordinates X	Co-ordinates Y	Log of Period	Median Magnitude
12321*	5576	10823	....	16.0	12333	16679	5093	0.613	16.65
12322	5934	9270	0.670	16.25	12334	16692	4836	0.458	16.6
12323	6048	10062	....	16.7:	12335	16728	6342	0.692	16.55
12324	6318	11850	Eclipsing	16.3 max.	12336	16776	5160	0.554	16.45
12325	9126	12012	0.596	16.35	12337	16878	5262	0.428	16.4
12326*	11494	7040	0.346	16.66	12338	16908	5166	0.614	15.95
12327	14928	5598	0.673	15.7	12339	16920	5490	0.511	16.25
12328	15330	5454	0.377	16.55	12340	16974	5310	0.421	16.65
12329	15465	5742	0.507	15.9	12341	16992	4578	0.492	16.20
12330	15678	5280	0.492	16.4	12342	17094	5106	0.708	15.85
12331†	16200	5328	Nova	13.3 max.	12343*	17100	5306	Irreg.	14.5 max.
12332	16440	6528	0.537	16.45					

* HV 12321 is No. 7, HV 12326 is No. 27, and HV 12343 is No. 55 in the list of 148 suspected variables (*Harvard Ann.*, Vol. **90**, No. 1, 1933).
† HV 12331 is Nova Doradus 1948 (these PROCEEDINGS, **40**, 366–368 1954; *Harvard Reprints*, No. 387).

The Period-Distribution Anomaly.—We have pointed out on earlier occasions that classical cepheids with periods less than 2 days are peculiarly scarce in the Large Cloud though abundant in the Small Cloud. Only eleven periods less than two and a half days are found among 321 classical cepheids along the bar:

HV No.	Period (Days)	Location	HV No.	Period (Days)	Location
12024	1.31289	Center	2750	2.3167	East
11994	1.7946	West	12032	2.375	Center
11997	1.92851	West	12328	2.3825	East
12030	2.075326	Center	12059	2.464	East
12326	2.2207	Center	12051	2.495	Center
5957	2.26491	East			

The classical cepheids in the Small Cloud reveal clearly the high correlation between position in the Cloud and period length, the longer periods predominating in the regions of highest star density. Earlier investigations of the distribution of the Large Cloud variables have shown this tendency much less clearly;[7] but now we find a concentration of the longer-period cepheids in the west end of the bar, where the apparent star density is somewhat less than in the central and eastern sections. The period distribution is shown in the accomanying tabulation, where we divide the bar into three main sections of comparable areas. Omitting from the west section a subsection that is actually "open territory" and not a part of the rich axis, we get the results labeled "West restricted," and immediately below we give the comparable data on period distribution for the Core of the Small Cloud.[8]

Region	Number of Variables	Mean Period (Days)	Median Period (Days)	PERCENTAGE >10 Days	PERCENTAGE <3 Days
East	113	4.481	4.061	4	13
Center	133	5.439	4.484	7	14
West	75	8.673	6.834	29	8
West restricted	37	11.725	9.711	49	0
Core of Small Cloud	34	11.274	6.915	38	26

The contrast in the values for the mean and median period lengths between west (or west restricted) and the east and central sections of the bar is conspicuous and undoubtedly of significance in the morphology and evolution of this irregular galaxy.

[The data for the Core of the Small Cloud show again (in the last two columns) the characteristic mentioned above, namely, that the Small Cloud variables with periods less than three days are abundant, even in the densest area, while in the west of the bar of the Large Cloud such periods are not found. Those appearing in the Core may be in large part superposed from the outlying parts of the Small Cloud.]

The explanation of the unexpected distribution of period lengths along the bar is not immediately at hand. In the section of the bar where the periods are long, there is no more gas-and-dust nebulosity than elsewhere, but the supergiant stars are relatively more numerous.

The Tilt of the Bar.—The long-period cepheids in our galactic system, with periods greater than two days, are strongly concentrated to low galactic latitudes. Very few, and they are mostly bright and near by, are in latitudes higher than $\pm 20°$. Open clusters are similarly concentrated to the galactic circle. In the Large Magellanic Cloud there are many such stars and clusters far from the bar and from the central mass of the Cloud. For example, at distances greater than two degrees from the geometrical center there are now on record 27 classical cepheids with periods longer than ten days and more than 50 open clusters.

Does this wide spread of massive stars suggest that the principal plane of the Large Magellanic Cloud is perpendicular to the line of sight? Or is the wide scattering merely indicative of the chaotic structure of this Magellanic galaxy? Perhaps the Cloud does not conform with the structural rules that apply to galactic concentrations in our own spiral and which are at least strongly suggested for the Andromeda Nebula.

We can test the first suggestion, the orientation of the plane, with the aid of the cepheid variables. For, if the Cloud is indeed flat (the resemblance to a barred spiral suggests such a form) and if the cepheid variables are concentrated to its plane, the cepheids should average to be systematically brighter in apparent magnitude on one side of the line of nodes than on the other, unless the Cloud's plane is essentially perpendicular to the line of sight.

We are not ready to make a definitive test for tilt of the whole Cloud, but we can readily test the inclination of the bar. If the period-luminosity curve for the variables at one end of the bar should be displaced vertically with respect to the curve for variables at the other end (magnitudes are ordinates), we can compute the average distances of the two ends and estimate the tilt. The standard relation used for this comparison is that derived some years ago for 137 variables in all parts of the Large Cloud.[9] For periods between two and forty days the empirical curve can be replaced with a straight line:

$$\dot{m} = 17.14 - 2.08 \log P.$$

In the accompanying tabulation we summarize the deviations from the standard period-luminosity curve (plot of logarithm of period against median *apparent* photographic magnitude). We find that the average deviations (third column) are positive for all parts of the bar, suggesting the eventual necessity of a revision of the above formula as a result of the revisions we are currently making in magnitude

Region	No. of Variables	Average Deviation	Corrected Deviation	Mean Error	δd in K.l.y.
East	69	+0.22	−0.08	±0.04	−6.5 ± 3.2
Near east	44	+ .22	− .08	± .05	−6.5 ± 4.0
East central	81	+ .27	− .03	± .03	−2.4 ± 2.4
West central	52	+ .43	+ .13	± .04	+10.5 ± 3.2
Near west	38	+ .26	− .04	± .05	−3.2 ± 4.0
West	37	+0.40	+0.10	±0.05	+8.1 ± 4.0
Total and means	321	+0.30	0.00	±0.043	0.0

sequences. Another contributing cause of the vertical displacement (Fig. 1) could be a background effect, since the variables lie in richer star fields than the master-sequence. (Or possibly this faintness of most of the variables along the bar is the result of greater intervening space absorption in that region than elsewhere in the Cloud.)

Before we derived the individual deviations for each of the 321 cepheids, the magnitude sequences used for photometry in the area of the bar were carefully examined

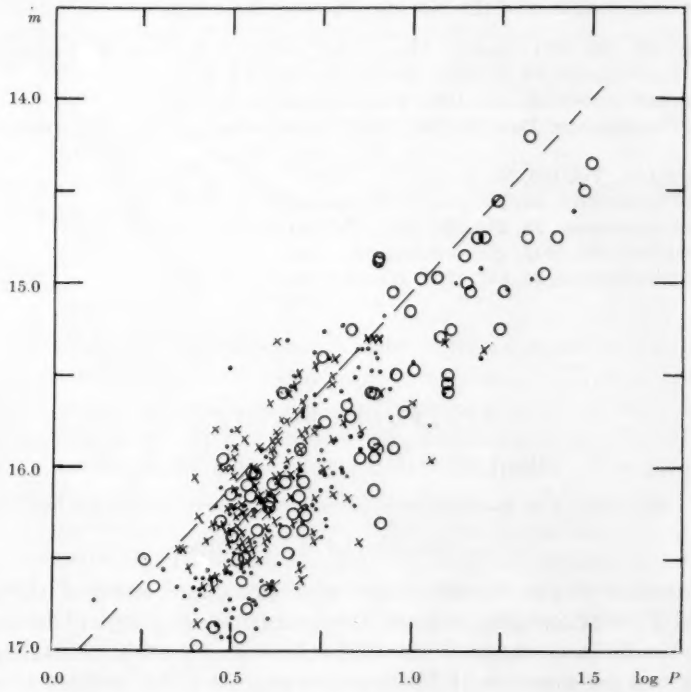

FIG. 1.—Period-magnitude array for the bar of the Large Magellanic Cloud.

and co-ordinated. A report on these sequences will appear in a later discussion of all the sequences.

The fourth column of the foregoing tabulation gives the mean deviations of the six fields adjusted for the displacement of their mean period-luminosity curves from the mean curve for the whole Cloud—a correction of +0.30 mag. The last column gives the computed differences in distance and their mean errors.

The differences in average distance shown by these data are of the same order as the mean errors. The suggested tilt would indicate that the east end may be slightly nearer the observer. The scatter about the mean period-luminosity curves is so large that the slight evidence of tilt cannot be taken seriously. Because of the scatter the method is not sufficiently sensitive to detect or disprove tilt in the relatively short bar. A more competent test is being made with the use of outlying fields.

The accompanying period-magnitude array illustrates the three principal results of our study of the bar, namely, apparent faintness of variables of all period lengths, peculiar distribution of supergiant long-periods cepheids, and the wide scatter of absolute magnitudes which conceals whatever axis tilt there may be. The open circles, representing variables in the west and near west, show the overlapping of its period-luminosity relation with those of the central ( ) and eastern (×) areas. Also, the systematic faintness of nearly all the 321 variables is clearly shown in the diagram, as well as the preference in the western area for the longer periods.

This part of our study of the Magellanic Clouds was supported by the American Philosophical Society and the National Science Foundation.

[1] *Science*, **120**, 788, 1954 (abstr.). Many observers have pointed out the resemblance.

[2] These PROCEEDINGS, **40**, 5, 1954; *Harvard Reprints*, No. 382.

[3] These PROCEEDINGS, **39**, 355, 1953; *Harvard Reprints*, No. 372.

[4] These PROCEEDINGS, **39**, 1162–1163, 1953; *Harvard Reprints*, No. 376; *Astron. J.*, **56**, 139–140, 1951.

[5] *Harvard Ann.*, Vol. **100**, No. 6, 1936.

[6] These PROCEEDINGS, **39**, 358–362, 1953; *Harvard Reprints*, No. 373.

[7] These PROCEEDINGS, **38**, 286–289, 1952; *Harvard Reprints*, No. 360.

[8] *Harvard Bull.* 916, 1942; *Harvard Bull.* 920, 1951.

[9] These PROCEEDINGS, **26**, 547, 1940; *Harvard Reprints*, No. 207.

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*ACETYL-CHYMOTRYPSIN**

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Communicated March 1, 1955

The reaction of the chymotrypsins with phosphate esters of the type of diisopropyl fluorophosphate, to form, for example, diisopropyl phosphoryl chymotrypsin, results in an inactive but still undenatured protein containing one phosphoryl group per molecule.[1] The reaction appears to be confined to some of the esterolytic enzymes,[2] including, however, the choline esterases. This is an interest-

ing case because the phosphorylation reaction appears to be the explanation of the toxicity of these phosphate esters and of their use as "nerve poisons" in insecticides. At the same time, the phosphorylation reaction possesses considerable fundamental interest, because it seems probable that the phosphoryl group is introduced at the active (hydrolytic) center of the enzyme.[3]

The introduction of a phosphoryl group appears to produce a particularly stable modification of chymotrypsin, which thus remains inhibited because the phosphorylated protein does not decompose under ordinary conditions. Yet there is no reason a priori why other groups should not associate themselves in like manner with the enzyme. On the contrary, there is evidence that the acetyl group may do so. Hartley and Kilby[4] have shown that when α-chymotrypsin reacts with *p*-nitrophenyl acetate, a rapid initial liberation of *p*-nitrophenol occurs in an essentially mole-for-mole proportion to the amount of enzyme present—a fact that they consider to be best explained by the acetylation of the enzyme at its active center. Since *p*-nitrophenyl acetate undergoes a slow catalytic decomposition subsequent to this initial rapid step, it is logical to picture the catalysis as a slow breakdown and rapid re-formation of the acetylated enzyme. A similar catalytic decomposition of *p*-nitrophenyl acetate (without any initial "burst" of activity) has been observed with insulin by Hartley and Kilby,[4] and by Dirks and Boyer[5] with native bovine serum albumin, heated egg albumin, and also with cysteine and glutathione.

In our laboratory we have experimented with this reaction in relatively acid media (pH 5.0–6.2) because under these conditions the initial "burst" of nitrophenol is still rapid, whereas the rate of spontaneous decomposition is negligible and that of the chymotryptic catalysis almost so. From the information thus obtained, a method was devised to isolate the reaction product and examine it in a state of rather high purity. It appears to be a monoacetyl derivative of chymotrypsin in which the acetyl group is unusually reactive. The substance is enzymatically inactive but, being quite unstable, is quickly and quantitatively converted back to active chymotrypsin. It appears to be distinct from the acetylated chymotrypsins prepared in other ways. The study is being continued, and further reports will be made later.

EXPERIMENTAL

Materials and Methods.—The nitrophenyl acetates were commercial products or else were synthesized and purified by the usual methods. α-Chymotrypsin was prepared from five-times-crystallized α-chymotrypsinogen and was itself crystallized three times thereafter. It was subsequently dialyzed against 0.001 *M* acetic acid, then against water, and finally dried *in vacuo* while frozen. The phosphorylated α-chymotrypsin was prepard as outlined by Jansen *et al.*,[6] but from *p,p*-dichlorophenyl chlorophosphate,[7] and was crystallized twice thereafter. It contained 0.11 per cent phosphorus. The esterolytic activity of chymotrypsin was measured with tyrosine ethyl ester at pH 6.2 by the method of titration at approximately constant pH.[8] A semimicro apparatus was used containing 3.0 ml. of 0.025 *M* substrate, to which about 1 *N* alkali was added from a microburette. The liberation of nitrophenol was measured in a Coleman junior colorimeter at 402 mμ. Hydroxamic acid formation was measured as outlined by Hestrin,[9] except that the reac-

tion was carried out at pH 6.4–6.5 and some changes were made in details of manipulation in order to suit our equipment. The method was calibrated with chymotrypsin present and with highly purified *p*-nitrophenyl acetate as a standard.

Liberation of Nitrophenol by α-Chymotrypsin.—In Table 1 the stoichiometry of the initial reaction between α-chymotrypsin and *o*-resp. *p*-nitrophenyl acetate is shown.[10] While rapid even at pH 5, the reaction is not instantaneous. Its speed is also pH-dependent. Larger quantities of protein reacted more slowly and less completely, possibly because the concentration of nitrophenyl acetate (which is decidedly limited by its solubility) was much less in relation to that of the protein.[11]

TABLE 1

Initial LIBERATION OF NITROPHENOL*

PROTEIN	WEIGHT (MG.)	MICROMOLES OF NITROPHENOL PER MICROMOLE OF PROTEIN†	
		ORTHO	PARA
α-Chymotrypsin	10.0	...	0.8
	11.3	0.9	...
	22.5	1.0	1.0
	45.0	0.9	...
	90.0	0.8	0.6
Acetyl chymotrypsin	25.0	...	0.18
α-Chymotrypsin in 30 per cent urea	25.0	...	0.06
p,p-Dichlorophenyl α-chymotrypsin	25.0	0.00	0.04
Chymotrypsinogen	25.0	0.00	...
	50.0	...	0.06

* Two milliliters of the nitrophenyl acetate to be used (0.25 *M* in ethanol) was stirred into 100 ml. of 0.066 *M* buffer (acetate, pH 5.0, when *o*-nitrophenyl acetate was used; phosphate, pH 6.2, when *p*-nitrophenyl acetate was used). Ten milliliters of the buffer was mixed at zero time with the protein previously dissolved in 1.0 ml. of water. The colorimeter was read at intervals (at 402 mμ) for at least 15 minutes. A control contained the buffer–nitrophenyl acetate solution, to which water had been added instead of protein solution. Other controls, containing the protein with or without known added quantities of the nitrophenol, showed no appreciable difference over the corresponding water blanks.

The initial reaction with α-chymotrypsin occurred in 2 minutes or less, except with large doses of protein. Thereafter, a straight line of small slope was observed representing a slow, steady liberation of the nitrophenol. The values given above were determined from this line by extrapolating it to zero time.

† The molecular weight of chymotrypsin is taken as 25,000.

In comparison with chymotrypsin, the results obtained with a number of related substances are shown, calculated in the same way (see Table 1). It may be noted that chymotrypsinogen, denatured chymotrypsin (in urea solution), and phosphorylated chymotrypsin all failed to give the characteristic rapid initial liberation of nitrophenol that was seen with the active enzyme. In this respect the situation parallels that observed when chymotrypsin reacts with one of the halogen phosphate esters. This similarity is further shown by the fact that the acetylated chymotrypsin is also enzymatically inactive.[12]

One micromole (25.0 mg.) of α-chymotrypsin was also reacted with a tenfold excess of ortho (at pH 5.0) and of para (at pH 6.2) nitrophenyl acetate in the small titration vessel ordinarily used for measuring esterolytic activity. No liberation of acid was observed other than the small amount to be expected from the slow and continuous decomposition that takes place during the steady state. Although these conditions are admittedly not ideal for the titration of a weak acid, the sensitivity of the apparatus was far greater than necessary to detect the liberation of one micromole of acetic acid. However, titratable acid was formed when the

acetylated protein isolated as described below was allowed to remain under conditions suitable for its reactivation. This formation of acid is being studied with some interest.

Preparation of Inhibited Chymotrypsin.—One gram of α-chymotrypsin was dissolved in about 200 ml. of water. The solution was then brought to pH 5.0, and 4 ml. of a solution of the nitrophenyl acetate in alcohol was stirred in rapidly. After an hour at room temperature the solution was adjusted to pH 3.8, chilled, and extracted 5–6 times with 50-ml. portions of cold chloroform. Following the extraction, about 0.2 gm. of washed "Norite-A" (wet) was stirred in and, after 5–10 minutes, removed by filtration. A second addition of "Norite" was then made and removed by filtration through very hard paper. (If any yellow tinge developed when a few drops of the filtrate were made alkaline, the adsorption was repeated again. This was seldom necessary.) The filtered solution was then evaporated *in vacuo* while frozen. The product should be white and completely soluble in water. As indicated by the hydroxamic acid test, specimens kept at 5° were found to have decomposed about 25 per cent in a month. Under these conditions decomposition yields some protein that is insoluble in water.

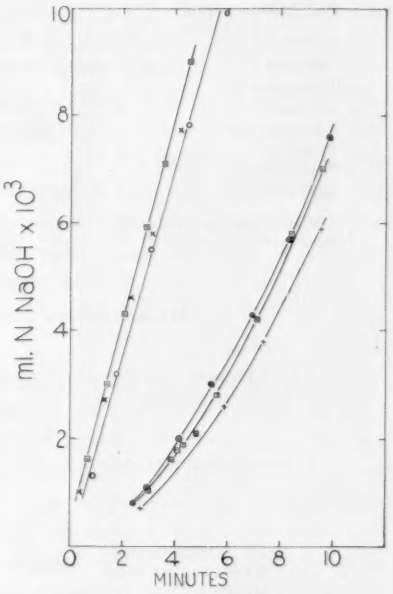

FIG. 1.—Rate of hydrolysis of tyrosine ethyl ester by acetylated chymotrypsin before (*right-hand curves*) and after (*left-hand curves*) reactivation; ○, product from *o*-nitrophenyl acetate; ⊡, product from *p*-nitrophenyl acetate; ×, product from 2,4-dinitrophenyl acetate.

Activity of the Isolated Material.—In the presence of a tenfold excess of nitrophenyl acetate (alcohol concentration *ca.* 1.5 per cent) the material had no demonstrable activity when 500–1,000 times the amount suitable for the assay of active chymotrypsin was employed. On the other hand, when introduced alone into the tyrosine ethyl ester solution (at pH 6.2), activity rapidly returned, as shown in Figure 1. The figure indicates, however, that the initial activity must be very small. The total activity that could be developed was measured after the lapse of 10 minutes in solution at pH 7.2. The assay curve after such reactivation is shown on the left side of the figure. The specific activity shown after exposure to pH 7.2 is given in Table 2 for several preparations. It was found to be very close to that of the original chymotrypsin from which the preparations were made.

It will be noted that the rapid reactivation of the protein in practically neutral solutions offers some technical difficulties to the determination of its activity *ab initio* on proteins. However, the proteolytic, esterolytic, and milk-clotting activities of chymotrypsin have never shown any indications of being separable properties. Accordingly, observations on the milk-clotting activity of the new protein were made with the thought that they might be sufficient to confirm the conclusion based on the inhibition of esterolysis. It was found, as shown in Table

3, that the milk-clotting activity of the new protein was very small compared to chymotrypsin, provided that short times of exposure were employed. It is also apparent that the length of exposure is more important for activity than is the initial quantity of enzyme. As might now be expected, the same enzyme solution after being neutralized again showed activity quite comparable to that of the original chymotrypsin. In its behavior toward milk the new protein appears to behave just as it does in esterolysis.

TABLE 2

HYDROXAMIC ACID FORMED BY THE ACETYLATED PROTEIN AND ITS SPECIFIC ACTIVITY AFTER "REACTIVATION"

Reagent	Hydroxamic Acid Formed (Moles per 25,000 Gm. of Protein)	Specific Activity* after Activation	Reagent	Hydroxamic Acid Formed (Moles per 25,000 Gm. of Protein)	Specific Activity* after Activation
o-Nitrophenyl acetate	1.1	0.029	*p*-Nitrophenyl acetate	1.0	0.034
o-Nitrophenyl acetate	1.1	.031	2,4-Dinitrophenyl acetate	1.0	.034
o-Nitrophenyl acetate	0.9	.035	2,4-Dinitrophenyl acetate	0.8	.035

* Milliequivalents of substrate decomposed per milligram of enzyme protein per minute. The original chymotrypsin had a specific activity of 0.036. The samples used were between 15 and 30 mg.

TABLE 3

MILK-CLOTTING ACTIVITY OF ACETYL-CHYMOTRYPSIN*

Enzyme	Weight (Mg.)	Clotting time (t) (Min.)	Relative Specific Activity ($1/t \times$ Mg.)
α-Chymotrypsin	0.50	0.6	3.3
	0.25	1.1	3.7
	0.10	2.5	4.0
Acetyl-chymotrypsin	2.00	0.7	0.72
	1.00	1.6	0.62
	0.50	2.0	1.00
	0.25	2.4	1.66
Same acetyl-chymotrypsin after reactivation†	0.45	0.8	2.8
	0.23	1.4	3.1
	0.09	3.0	3.7

* Pasteurized skimmed milk (pH 6.4) was made 0.01 M with respect to added $CaCl_2$. Various amounts of enzyme were dissolved in 1.0 ml. of water and mixed with 10 ml. of the milk at 40°. All measurements were made within a few hours on the same sample of milk and are therefore comparable. Within reasonable limits the clotting time of active chymotrypsin is inversely proportional to the dose of active enzyme; thus $1/t \times$ mg. of enzyme should be approximately a constant whose numerical value is larger the more active the enzyme. The method is not highly accurate (in our hands).

† The water solution of the acetylated enzyme at pH 3.8 was brought to pH 7.2 and allowed to stand about 12 minutes at room temperature.

Other Properties.—Table 2 shows the results of the reaction with hydroxylamine at pH 6.4–6.5, wherein essentially one mole of hydroxamic acid was formed per mole of protein. From this it may be concluded that the isolated substance is largely a mono-substituted derivative of the enzyme protein in which the substituent group appears to be sufficiently reactive to qualify as a substance of "high energy" in the biochemists' sense of the term. However, the presence of acetic acid as such has not been shown as yet.

In view of the foregoing observations, it seems reasonable to consider the isolated material as a fairly pure substance. Nevertheless, it is also nothing more than the

total residue of an evaporation. An attempt to meet this objection has been made. It was found that the protein, after being dissolved in water and precipitated with ammonium sulfate, had the same hydroxamic acid value as at the start, provided that the exposure to ammonium sulfate was not very long.

An experiment that showed this was made as follows: 25–30 mg. of protein was dissolved in 2 ml. of water, and an equal volume of saturated ammonium sulfate solution (adjusted to pH 3.8 as measured when diluted to half-saturation) was added. Further additions during the next half-hour brought the saturation to 60–65 per cent. When the precipitate was centrifuged down at this time, it was found to have the same hydroxamic acid value as the original protein. (Very little protein remained in solution.) Longer suspension in ammonium sulfate solution led to crystal formation (in 2–3 hours), which appeared to be practically complete the next day. By this time, however, the precipitated protein had lost between half and two-thirds its original capacity to form a hydroxamic acid.

The experiment probably indicates that mixed crystals of active and inhibited chymotrypsin are formed. However, chymotrypsin crystallizes neither rapidly nor well at such a high salt concentration, as was indeed shown by the accompanying controls of α-chymotrypsin, which did not crystallize at all. Moreover, the crystals were quite different from those usually formed by α-chymotrypsin. This form persisted through a recrystallization. The main point, however, is that the substance can be precipitated from an aqueous solution.

DISCUSSION

The acetylation of chymotrypsin follows closely the pattern of its phosphorylation. Acetyl-chymotrypsin, however, by reason of its instability seems to offer still less hope of locating the group to which it is attached. It is evident that the enzyme molecule offers to both reagents one particularly active group which is not available in the precursor protein or in the denatured protein. It seems highly probable that the same group is involved in both instances. If so, the rapid and complete reactivation of the acetyl derivative is good evidence that the group in question is indeed a part of the essential active center of the enzyme.

In all likelihood there are many enzymes (probably esterases) that can form derivatives analogous to acetyl-chymotrypsin; their probable role in the transfer of acetyl groups becomes a matter of interest. For instance, the presence of chymotrypsin in an acetylating system would suggest that it might readily accept acetyl and as readily release it as acetate—in other words, "run down" the system. Preliminary experiments with the acetylation of sulfanilamide by the pigeon liver[13] system of Kaplan and Lipmann[14] have indeed indicated that the presence of chymotrypsin does decidedly reduce the amount of sulfamide acetylated, principally by reducing the active life of the acetylating system.

The ready formation of acetyl-chymotrypsin from a nitrophenyl acetate also strongly suggests the fulfilment of a substrate requirement for that enzyme. Should an enzyme of similar substrate requirement occur in a respiring tissue, the well-known effect of nitrophenols on the uncoupling of phosphorylation could be plausibly explained on the hypothesis that the nitrophenol might well be acetylated.

It has been shown by Dirks and Boyer[5] that sulfhydryl compounds are good catalysts for the hydrolysis of *p*-nitrophenyl acetate. The mechanism of this ca-

talysis appears to be similar to that ascribed to chymotrypsin, namely, through the formation of an acetyl compound.[15] It is therefore intriguing to speculate on the possibility that the reactive group of chymotrypsin might be a sulfhydryl group. This is unfortunately a case in which negative results cannot constitute a positive answer; all that can be said at present is that our efforts to demonstrate the existence of an essential sulfhydryl group in chymotrypsin have failed so consistently that we are beginning to doubt its existence.

SUMMARY

An acetyl derivative of chymotrypsin has been isolated from the reaction mixture of α-chymotrypsin with *o*-nitrophenyl acetate, *p*-nitrophenyl acetate, and 2,4-dinitrophenyl acetate. The substance contains one highly reactive acetyl group per molecule of protein, as shown by the ability to form a hydroxamic acid at pH 6.5. It is inactive as an esterase and does not clot milk. Although easily decomposed in alkaline solutions to the active enzyme, it has been precipitated in acid solutions by ammonium sulfate without loss of acetyl.

The existence of this substance leads to speculations on its possible effect in physiological acetylating systems and also on the mode of action of nitrophenols therein.

* Journal Paper No. 850, Purdue University Agricultural Experiment Station, Lafayette, Ind.

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[10] Similar qualitative results were obtained with 2,4-dinitrophenyl acetate but were not studied quantitatively.

[11] The concentration of alcohol in the system (*ca.* 2 per cent) was not regarded as objectionable, though concentrations higher than this have been carefully avoided.

[12] No initial "burst" of nitrophenol was found in the decomposition of the acetates when catalyzed by cysteine, glutathione, or serum albumin, although the catalytic effect of these substances is quite definite.

[13] As pigeon livers are not commonly available here, this experiment could not have been made but for the wholehearted co-operation of the West Lafayette Police Department.

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[15] To be published later.

THE MODE OF GROWTH OF EPIDERMAL CELLS OF THE AVENA COLEOPTILE

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From the dense texture of cellulose microfibrils in electron micrographs of the primary wall, Mühlethaler[1] concluded that parenchyma and epidermal cells of the oat coleoptile could elongate only at their tips or ends. This concept of "tip growth" of the primary walls of *Avena* cells derives only from microscopic observations on fixed and extracted material, yet has achieved currency. The present observations on marked, living cells attempt a direct test of the reality of tip growth for such cells.

Small areas of the living coleoptile, and in favorable cases single epidermal cells, were marked by dusting with a fine CuO powder. These black particles readily adhere, can be winnowed to suitably small dimensions during application, are insoluble and nontoxic for at least a period of days, and photograph well. Generally, sections 3 mm. long were cut from the upper third of 3-day-old coleoptiles and mounted on an uncovered slide in a shallow drop of water containing 1 per cent sucrose and 10 mg/l indoleacetic acid. After light dusting with CuO powder, the spacings of particular particles were measured with a filar micrometer under low or high power of the microscope, or the field of interest was photographed. Best optical conditions result if the leaf is removed from the center of the hollow coleoptile. The isolated sections continued to grow on the slide, sometimes doubling in length in 24 hours; their top surfaces were in air, and curvatures were inapparent under these conditions. Between measurements the slides were returned to a moist chamber, and water lost by evaporation was replaced if necessary.

Growth of a small area of the outer epidermis may be studied by comparing the relative positions of CuO particles as found on successive photographs. On each negative some one identifiable marker is taken as an arbitrary origin for measurements of the distance of other markers along the coleoptile's axis. Let the initial distance from this origin for any marker be D_0, and let the corresponding distance after time t be D_t. Figure 1 shows the positions of seventeen separate markers before and after a 2.5-hour period of growth, D_0 being plotted against D_t in each case. The points fall satisfactorily on a single straight line having a slope of 1.10, which thus represents a 10 per cent growth in length of this area in 2.5 hours. Two or more markers on any nongrowing part of the wall would be connected by a line having a slope of 1.00, denoting the absence of growth between them. Unlike the elegant measurements of Sinnott and Bloch,[2] optical conditions here do not permit mapping of this area in terms of individually identified and delimited cells. Yet there is no evidence in the plot for nongrowing regions; instead, growth appears to be very uniformly distributed over the area in question, which was initially less than 0.5 mm.^{2}. According to Avery and Burkholder,[3] the average length of an outer epidermal cell in a 14-mm. coleoptile is about 0.5 mm.

In particular cases it was possible to find a series of CuO markers distributed along a single cell and to follow under high power the increase in their spacings.

Table 1 gives the data for distances between five markers in series on a single inner epidermal cell, exposed to access by cutting the coleoptile section lengthwise. In this case the initial distances between adjacent markers range from 10 to 40 μ, measured to the nearest micron, and the increase in each after 22 hours is between 20 and 25 per cent. No data have been obtained that show large differences in growth rate, such as might be expected on the basis of growth wholly restricted to the tips of cells.

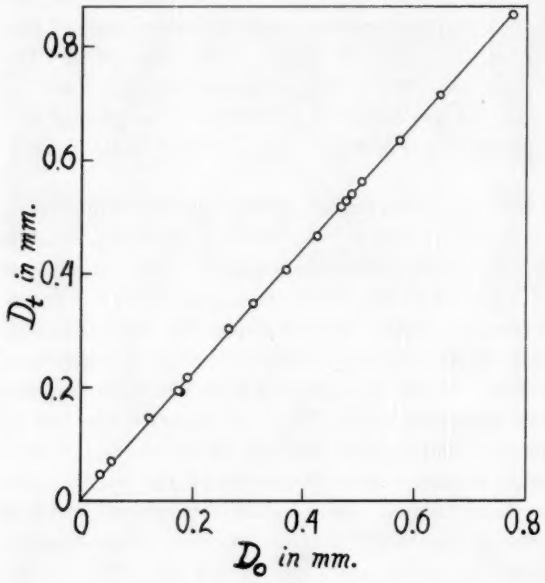

FIG. 1.—Initial position (D_0) plotted against position after 2.5 hours (D_t) of seventeen markers on a small area of epidermis, measured from an arbitrary origin. The slope of the line is 1.10. (See text.)

TABLE 1

DISTANCES BETWEEN 5 MARKERS IN SERIES ON A SINGLE INNER EPIDERMAL CELL, INITIALLY AND AFTER 22 HOURS OF GROWTH

Initial Spacing (μ)	Spacing after 22 Hours (μ)	Change (Per Cent)
10	12	20
40	50	25
10	12	20
24	29	21

If the assumptions underlying the present marking experiments are valid, there is no evidence from growth studies that *Avena* coleoptile cells of the inner or outer epidermal layers show "tip growth." The outer epidermal cells are known to elongate without cell division and to reach a length as great as 150 times their embryonic length. This enormous extension of their primary longitudinal walls appears to be achieved by growth rather uniformly distributed along the length of the cell, in spite of impediments suggested by the density and texture of microfibrillar arrangements revealed by electron microscopy. The presumptive mode of surface enlargement of the primary wall need not be different in kind from that occurring in the growing regions of plant hairs studied by Roelofsen[4] and termed

by him "multi-net growth." The tip growth of cells in a pavement type of tissue would indeed be difficult to reconcile with the findings of Sinnott and Bloch[2] on cell interrelationships in the roots of grasses.

Summary.—Study of CuO markers on the surface of epidermal cells of the *Avena* coleoptile indicates that during growth each cell is increasing in length at a rate that is rather uniformly distributed along the cell. No evidence is found for the idea that such cells grow exclusively at their tips or ends.

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THE CRYSTAL STRUCTURE AND TWINNING OF Co_2S_3

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Introduction.—The structure of Co_2S_3 is apparently unknown. It was first made by von Fellenberg,[1] who obtained it by heating together unspecified proportions of $CoCO_3$, flowers of sulfur, and K_2CO_3. Leaching of the mass with water left thin, graphite-gray, shiny metallic plates. Von Fellenberg analyzed these by roasting 0.229 gm. alternately in a stream of air and hydrogen until the resulting cobalt metal had a constant weight. The loss of weight was taken to be S, and the analysis was compared with the composition of ideal Co_2S_3 as follows:

	Found	Calculated for Co_2S_3
Co	55.45%	55.01%
S	44.55	44.99
	100.00	100.00

Schneider[2] repeated von Fellenberg's synthesis, using the following charge: 1 part by weight Co (or 2 parts $CoCl_2$), 18–24 parts by weight K_2CO_3, and 18–24 parts by weight S. He noted that, in addition to the hexagonal, whitish-gray metallic flakes, the preparation always contained a black, somewhat ill-defined crystalline powder. The admixed powder could be avoided by using the following charge: 1 part by weight $CoCl_2$, 6 parts by weight Na_2CO_3, and 6 parts by weight S. These were melted together for 10 minutes at bright red heat. The product was analyzed by determining the Co as metallic cobalt and the S as barium sulfate. The results of the analysis were as follows:

	Found	Calculated for Co_2S_3*
Co	55.45%	55.55%
S	44.45	44.45
	99.90	100.00

* Based upon atomic weights, Co = 60, S = 32.

Our own material was made by following approximately Schneider's first synthesis. Specifically, 1 part by weight Co, 6 parts by weight K_2CO_3, and 6 parts by weight S were melted together and held at about 900° C. for an hour. When the resulting melt was leached with water, brilliant whitish-gray metallic plates, some of which had a hexagonal outline (Fig. 1) were found admixed with a black powder. We separated 0.120 gm. of plates by hand picking. These gave the analyses shown in the accompanying tabulation. This confirms that the plates have substantially the composition Co_2S_3 assigned to them by Schneider. The density of the plates, as determined using a Berman balance, was 4.85 gm/cc.

	Found (Per Cent)	Per Cent Calculated for		
		CoS_2	Co_2S_3	Co_3S_4
Co (electrolytic)	54.58	47.89	55.06	57.96
S (Carius, gravimetric)	44.69*	52.11	44.94	42.04
	99.27	100.00	100.00	100.00

* Average of three values: 44.76, 44.58, and 44.74 per cent.

Identification of the Powder.—The black powder was found to be composed of tiny octahedra. One of these was examined by rotation and by deJong-Bouman photographs and was found to be isometric, with $a = 5.56$ Å, and to belong to space group $P2_1/a\ \bar{3}$. This symmetry and the length of the cell edge made it fairly certain that the crystal had a pyrite-type structure, and this was confirmed by taking corresponding deJong photographs of the mineral pyrite. The photographs of pyrite and the unknown showed a similar distribution of intensities. It is therefore evident that the powder is composed of CoS_2 crystals of pyrite type. The CoS_2 structure has been investigated by deJong and Willems,[3] who found $a = 5.64$ Å, and by Lundquist and Westgren,[4] who found $a = 5.524$ Å. Our own measurement of $a = 5.56$ Å is close to that of Lundquist and Westgren, and it seems certain that deJong and Willems' value is considerably too high.

Investigation of the Co_2S_3 Plates.—The plates were studied in detail by the precession method. They proved to have trigonal symmetry and were based upon a primitive cell having the following dimensions: $a = 6.68$ Å, $c = 16.35$ Å, and $V = 632$ Å^3. The diffraction symmetry can be represented by the symbol $3mP3_1--$.

The dimensions of the cell bear a close resemblance to comparable compounds said to have the nickel arsenide structure,[5] specifically, CoS and FeS. The comparison is given in the following table:

	Hexagonal Plates	CoS	FeS
a	2×3.34	3.37 Å	3.43 Å
c	3×5.45	5.14 Å	5.68 Å
Z	....	2 CoS	2 FeS

This comparison suggested that the plates might bear a structural relation to CoS. If this were so, the multiplicity of the superstructure would be $2 \times 2 \times 3 = 12$, so that the compositions of the plates ought to be related to the content of 12 cells of CoS. Since the CoS structure has 2 CoS per cell, the composition of the plates should be related to 24 Co and 24 S atoms. The simplest relationship should be that the 24 S atoms are constant for both structures and that some of the 24 Co atoms are omitted in the plates in a regular manner in such a way as to give rise to a superstructure of the observed multiplicity.

Neither the cell density nor the superstructure assumption just suggested is in simple accordance with the supposed formula Co_2S_3. The density requires 8.62

FIG. 1.—Hexagonal plates of Co_2S_3. ×30.

Co_2S_3 per cell, which is irrational. If, however, this is accepted as signifying 9 Co_2S_3 per cell, then there are 18 Co and 27 S per cell, but this is no longer consistent with the superstructure based upon 24 S atoms.

A more acceptable explanation of both density and superstructure can be based upon the composition Co_3S_4. The observed density requires 6.05 Co_3S_4 per cell, which is sufficiently close to an integer to be acceptable. The cell would contain 18 Co and 24 S. This is obviously consistent with a superstructure based upon 24 S atoms. Furthermore, the 18 Co atoms are distributable into three equivalent layers, as required by a 3-fold screw axis. Under these circumstances the plates would have the following characteristics: $a = 6.68$ Å, $c = 16.35$ Å, $V = 632$ Å^3, $d = 4.85$ gm/cc, and $Z = 6$ $Co_{3-x}S_4$ per cell.

If the plates indeed are related to the composition Co_3S_4, then it is important to investigate the relation of their structure to that of linnaeite,[6] Co_3S_4. Linnaeite is isometric, with cell edge $a = 9.398$ Å, space group $F\ 4_1/d\ \bar{3}\ 2/m$, and density 4.85 gm/cc. If an isometric crystal is referred to hexagonal axes, the relation of the hexagonal cell edges to isometric cell edge a is $A = \sqrt{2}a/2$, $C = \sqrt{3}a$. The comparison of cell dimensions for linnaeite and the hexagonal plates referred to the same axes is given in Table 1. The close agreement between these values is striking. In view of this, we took powder photographs of the hexagonal plates and also of linnaeite and found them to be substantially indistinguishable.

TABLE 1

COMPARISON OF CHARACTERISTICS OF Co_2S_3 PLATES WITH LINNAEITE

REFERRED TO HEXAGONAL AXES

		Linnaeite	Co_2S_3 Plates
A		6.64 Å	6.68 Å
C		16.26 Å	16.35 Å
Cell type		R	R
Volume, V	$^3/_4 \times 830 =$	622 Å^3	632 Å^3

REFERRED TO CUBIC AXES

		Linnaeite		Co_2S_3 Plates
a		9.398 Å	$\sqrt[3]{V} =$	9.44 Å
Cell type		F		F
Volume, V		830 Å^3	$^4/_3 \times 632 =$	841 Å^3
Cell contents		8 Co_3S_4:		
	$\Rightarrow$	24 Co	$\Rightarrow$	$21^1/_3$ Co
		32 S		32 S

This was a surprising result. The crystallographic identity of the hexagonal plates and an isometric crystal would be reasonable only if the plates were actually thin spinel twins. To check this possibility, the reciprocal-lattice levels recorded for the hexagonal plates were carefully studied, and it was found that they indeed contained lattice points having a distribution characteristic of the twin in question.[7] Specifically, the observed points could be reproduced by taking levels normal to a 3-fold axis of the reciprocal lattice for a face-centered isometric cell and superposing this pattern on itself after a 180° rotation about the 3-fold axis. The diffraction symmetry of the plates is also consistent with the theory of the symmetry of twins.[7] From this it seemed certain that each hexagonal plate was a set of spinel twins of the linnaeite structure and that the habit of each individual is a slice parallel to (111) through an octahedron. To check this, precession photographs were made of a single crystal of linnaeite taking [111] as the precession axis. The intensity distribution for each level of linnaeite matched that of the

corresponding level of a hexagonal plate of Co_2S_3, due allowance being made for twinning in the latter.

Conclusions.—The analyses of previous investigators, and the new analysis included in this paper, establish the composition of the hexagonal plates as Co_2S_3. On the other hand, the correspondence in dimensions of the Co_2S_3 cell with that of linnaeite, the substantially identical powder patterns of Co_2S_3 and linnaeite, and the correspondence in intensities in the precession photographs establish the structure of Co_2S_3 as spinel type, with some cobalt atoms missing.

Specifically, Co_2S_3 is isometric, with $a = 9.44$ Å, space group $F\ 4_1/d\ \bar{3}\ 2/m$, and cell contents $Co_{21\frac{1}{3}}S_{32}$. The 32 S atoms occupy equipoint 32*e*, while the $21^1/_3$ Co atoms are statistically distributed over the total of 24 positions in the two equipoints 8*a* and 16*d*. The structure of Co_2S_3 can be described as the spinel-type structure in which $2^2/_3$ Co atoms per cell are missing, and the remaining $21^1/_3$ Co atoms satistically occupy the locations of the 24 Co atoms in Co_3S_4. The relation between Co_2S_3 and Co_3S_4, therefore, is similar to the relation between isometric Fe_2O_3 (maghemite) and Fe_3O_4 (magnetite).

It is noteworthy that in solids prepared by the solidification of melts[4] in the system Co-S, the Co_3S_4 structure type extends over the range 54–66 atomic per cent S. This corresponds to a composition range of $Co_{3.4}S_4$ to $Co_{2.06}S_4$ and therefore includes the composition Co_2S_3. In such melt-grown solids, the composition Co_2S_3 has no discrete existence. Nevertheless, the fact that the particular composition Co_2S_3 has been repeatedly found indicates that it represents a low-energy composition. Accordingly, subsolidus phase-diagram investigations of the system Co-S may be expected to find a singularity at this composition.

A most unusual feature of the Co_2S_3 crystals is that every crystal of several large batches examined had the form of a thin plate, many of these plates displaying a hexagonal outline (Fig. 1). Every such plate was evidently a twinned "crystal." No single crystals occurred. The universal occurrence of twins and the failure of occurrence of single individuals is unusual.

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THE STRUCTURE OF LIQUID HELIUM*

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In liquid helium at temperatures in the neighborhood of absolute zero, macroscopic manifestations of quantum mechanics are clearly evident. Quite apart from the HeI–HeII transition and the phenomenon of superfluidity, we observe in the highly expanded structure of the liquid an effect which is certainly associated with the high zero-point energy of the atoms. It is this effect which we propose to treat in the present article. Although we shall not undertake an a priori calculation of the zero-point kinetic energy of the liquid, we shall show how this quantity may be determined from its density and how it may be employed to determine the radial distribution function and the intensity of X-rays scattered by the liquid.

We shall present only a brief sketch of the basic theory here, reserving detailed treatment for a later article. We start with the Wigner[1] representation of the density matrix $f^{(N)}(\vec{R}_1, \ldots, \vec{R}_N, \vec{p}_1, \ldots, \vec{p}_N)$ in the phase space $(\vec{R}_1, \ldots, \vec{R}_N, \vec{p}_1, \ldots, \vec{p}_N)$ of a system of N helium atoms, the properties of which have been investigated by Irving and Zwanzig.[2] By integration over the phase space of $N - n$ atoms, contracted Wigner functions,

$$f^{(n)}(\vec{R}_1, \ldots, \vec{R}_n, \vec{p}_1, \ldots, \vec{p}_n) = \int \cdots \int f^{(N)} \prod_{k=n+1}^{N} d^3\vec{R}_k \, d^3\vec{p}_k, \tag{1}$$

in the phase space of subsets of n atoms may be constructed. When these functions are integrated over momentum space, they yield average number densities $\rho^{(n)}$ in the configuration space of the subset n.

$$\rho^{(n)}(R_1, \ldots, \vec{R}_n) = \int \cdots \int f^{(n)}(\vec{R}, \vec{p}) \prod_{k=1}^{n} d^3\vec{p}_k. \tag{2}$$

Of fundamental importance in the set of average number densities is the pair density $\rho^{(2)}(R_{12})$, which is related to the radial distribution function $g(R)$ in the following manner:

$$\rho^{(2)}(R_{12}) = \left(\frac{N}{v}\right)^2 g(R_{12}), \tag{3}$$

where N/v is the constant average density $\rho^{(1)}$ of the fluid. The equation of state of the liquid is determined by the radial distribution function $g(R)$ and the potential of interatomic force $V(R)$ in the form

$$\begin{aligned} Pv &= Nk\tau - \frac{2\pi N^2}{3v} \int_0^\infty R^3 \frac{dV}{dR} g(R) \, dR, \\ k\tau &= \frac{1}{3\rho^{(1)}} \int \frac{P^2}{m} f^{(1)}(\vec{R}, \vec{p}) \, d^3\vec{p}, \end{aligned} \tag{4}$$

where P is the pressure and $k\tau$ is equal to two-thirds of the mean kinetic energy per molecule of the fluid. For classical fluids τ is equal to the thermodynamic temperature T, but for quantum fluids it may be considerably greater than T, due to a high zero-point kinetic energy. We shall refer to τ as the "zero-point temperature" of the fluid. In a similar manner, the molal internal energy, E, of the fluid may be expressed as

$$E = \frac{3}{2} Nk\tau + \frac{2\pi N^2}{v} \int_0^\infty R^2 V(R) g(R)\, dR. \tag{5}$$

Multiplication of both sides of Irving and Zwanzig's[2] equation(2.18) by the momentum $\vec{p}_i$ of any member i of a set of n atoms and integration over all momentum space and over all configuration space except that of the subset n yields the following set of integrodifferential equations for average number densities $\rho^{(n)}$ in the configuration space of subsets n of the entire system of N atoms:

$$\begin{aligned} \nabla_{\vec{R}_i} \cdot (k\tau_i^{(n)} \rho^{(n)}) = &-\left(\sum_{\substack{k=1 \\ \neq i}}^{N} \nabla_{\vec{R}_i} V(R_{ik}) \right) \rho^{(n)} \\ &+ \int \nabla_{\vec{R}_{n+1}} V(|\vec{R}_{n+1} - \vec{R}_n|) \rho^{(n+1)}\, d^3\vec{R}_{n+1}, \\ &\qquad i = 1, \ldots, n; \qquad n = 1, \ldots N-1, \end{aligned} \tag{6}$$

$$k\tau_i^{(n)} = \int \cdots \int \frac{\vec{p}_i \vec{p}_i}{m} \frac{f^{(n)}}{p^{(n)}} \prod_{k=1}^{n} d^3\vec{p}_k.$$

Similar equations have also been derived by Born and Green,[3] using a general representation of the density matrix. The quantities $\rho^{(n)} k\tau_i^{(n)}$ represent the components of the average momentum current density tensor in the configuration space of subsets of n atoms. For classical systems in thermodynamic equilibrium, all $\tau_i^{(n)}$ are equal to $\mathbf{1}_i T$, where T is the thermodynamic temperature and $\mathbf{1}_i$ is the unit dyad in the configuration space of molecule i. For quantum-mechanical fluids the $\tau_i^{(n)}$ may depart widely from the classical value and will in general be functions of position. Their calculation would involve the complete solution of the quantum-statistical problem and the determination of the density matrix. The effect of statistics on the densities $\rho^{(n)}$ is implicit through its effect on the tensors $\tau^{(n)}$.

If we introduce the approximation of superposition in the configuration space of molecular triplets, employed by Kirkwood, Lewinson, and Alder[4] in the determination of the classical radial distribution function of fluids, the system of integral equations, equations (6), for a system of uniform average singlet density, may be closed in the form of a single integrodifferential equation for the radial distribution function $g(R_{12})$:

$$\begin{aligned} \nabla_{\vec{R}_{12}} \cdot (k\tau^{(2)} g(R_{12})) = &-(\nabla_{\vec{R}_{12}} V(R_{12})) g(R_{12}) \\ &+ g(R_{12}) \frac{N}{v} \int \nabla_{\vec{R}_{23}} V(R_{23}) g(R_{23}) g(R_{13})\, d^3\vec{R}_3, \end{aligned} \tag{7}$$

$$\tau^{(2)}(R_{12}) = \tau\mathbf{1} + (\tau^{(2)} - \tau\mathbf{1}).$$

If the tensor $\tau^{(2)}$ is approximated by its asymptotic value $\tau\mathbf{1}$ for large interatomic distances, equation (7) becomes,

$$k\tau\nabla_{\vec{R}_{12}}g(R_{12}) = -(\nabla_{\vec{R}_{12}}V(R_{12}))g(R_{12}) + g(R_{12})\frac{N}{v}\int \nabla_{\vec{R}_{23}}V(R_{23})g(R_{23})g(R_{13})\, d^3\vec{R}_3, \quad (8)$$

which is identical with the equation for the radial distribution function for classical fluids, except that, as in the equation of state, equation (4), the thermodynamic temperature T is replaced by the zero-point temperature τ. The classical equation has been solved numerically by Kirkwood, Lewinson, and Alder[4] for the Lennard-Jones potential of interatomic force:

$$V(R) = 4\epsilon\left\{\left(\frac{a}{R}\right)^{12} - \left(\frac{a}{R}\right)^{6}\right\}. \quad (9)$$

If the zero-point temperature τ is known as a function of the temperature and density of the fluid, the classical solutions, $g(R, \tau)$, describe the structure of a quantum fluid, subject to the validity of the approximations underlying equation (8).

In the absence of an adequate theoretical method for determining τ, we may employ the equation of state to determine this quantity, in the form

$$\tau = \frac{2\pi N}{3kv}\int_0^\infty R^3\frac{dV}{dR}\, g(R, \tau, v)\, dR + \frac{Pv}{Nk}. \quad (10)$$

With an experimental value of the molal volume $v(T, P)$ of the quantum fluid, τ may be calculated as a function of thermodynamic temperature and density. For liquid He^4 at 1.4° K. under its own vapor pressure, the tabulated integral[4] of equation (10) leads to a value of τ of 12.8°, and τ exhibits little sensitivity to temperature below the λ-point. For this purpose the de Boer[5] constants $\epsilon = 14.10 \times 10^{-16}$ ergs, $a = 2.56$ Å, were employed in the Lennard-Jones potential, equation (9). The zero-point temperature τ may also be estimated from the experimental energy of vaporization with the use of equation (5) and the classical tables of the internal-energy integral.[4] The latter value is 13°, which agrees rather well with the equation-of-state estimate.

Employing the reduced tables of Kirkwood, Lewinson, and Alder,[4] the de Boer potential constants,[5] and a zero-point temperature of 12.8°, we have calculated the radial distribution function $g(R)$ for liquid He^4 at a thermodynamic temperature of 1.4° under its own vapor pressure. The theoretical radial distribution function is presented as a function of interatomic distance in Table 1 and Figure 1. The structure is abnormally expanded as compared with a classical fluid at the same temperature, due to the high zero-point energy of the quantum fluid. The integral under the first peak of $g(R)$ leads to an estimate of six for the number of nearest neighbors of an atom, in good agreement with Keesom's[6] estimate from early X-ray scattering data. The more recent experimental radial distribution function of Reekie and Hutchinson[7] yields four nearest neighbors at 2.06° K., as well as an additional six next-nearest neighbors arising from a satellite peak on the outer shoulder of the first maximum. The theoretical radial distribution function does not show this satellite peak.

The intensity of X-ray scattering by a monatomic liquid is related to the radial distribution function and the scattering angle θ by the familiar relation

$$\frac{I(\theta)}{I_0} = 1 + \frac{4\pi N}{vs}\int_0^\infty R \sin(sR)[g(R) - 1]\,dR, \; s = \frac{4\pi}{\lambda}\sin\left(\frac{\theta}{2}\right), \quad (11)$$

where λ is the wave length and I_0 is the scattering intensity produced by a single atom in the gas phase. We have calculated the Fourier transform of the theoretical radial distribution function as a function of the reduced variable σ, equal to $as/4\pi$, where, again, a is the appropriate Lennard-Jones parameter. The theoretical

TABLE 1

THEORETICAL RADIAL DISTRIBUTION FUNCTION FOR $\tau = 12.8°$

x	$g(x) - 1$	x	$g(x) - 1$	x	$g(x) - 1$	x	$g(x) - 1$
1.00	0.094	1.52	0.018	2.04	−0.010	2.56	0.009
1.04	.591	1.56	− .016	2.08	.009	2.60	.004
1.08	.930	1.60	− .042	2.12	.025	2.64	.001
1.12	.993	1.64	− .060	2.16	.037	2.68	− .002
1.16	.904	1.68	− .074	2.20	.044	2.72	− .004
1.20	.760	1.72	− .082	2.24	.047	2.76	− .006
1.24	.608	1.76	− .087	2.28	.046	2.80	− .007
1.28	.470	1.80	− .088	2.32	.044	2.84	− .008
1.32	.352	1.84	− .081	2.36	.039	2.88	− .008
1.36	.254	1.88	− .074	2.40	.033	2.92	− .008
1.40	.173	1.92	− .064	2.44	.026	2.96	− .007
1.44	.108	1.96	− .050	2.48	.021	3.00	−0.007
1.48	0.057	2.00	−0.030	2.52	0.014		

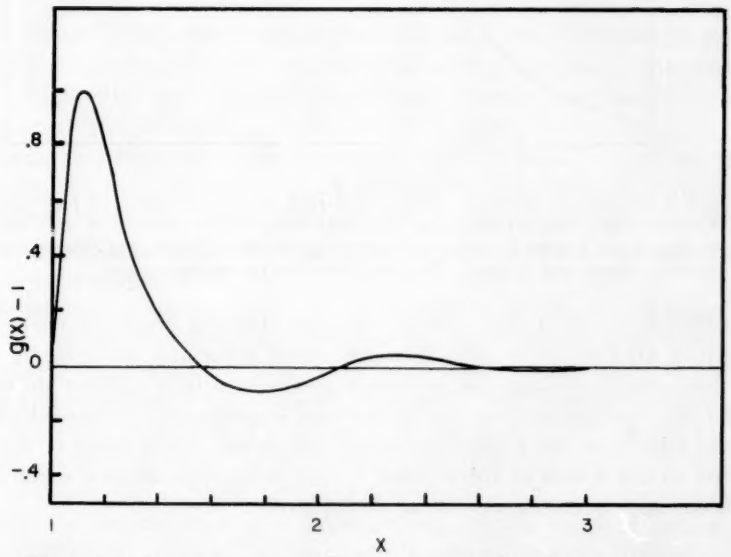

FIG. 1.—$g(x) - 1$ versus x. Calculated radial distribution function for liquid helium at 1.4° K. versus reduced distance, $x = R/a$.

scattering intensity is compared with recent measurements of Gordon, Daunt, and Shaw[8] in Figure 2. In the case of the experimental curve, the reduced variable σ is set equal to $a_0 s/4\pi$, with a_0 so chosen as to place the first peaks of the theoretical and experimental curves at the same value of σ. The value of a_0, equal to 2.99, is somewhat larger than the Lennard-Jones parameter a. The discrepancy between a and a_0 is doubtless attributable to deficiencies of certain of the approximations of the

theory. However, apart from this discrepancy the agreement between theory and experiment is remarkably good.

For the comparison of our theoretical $g(R)$ with experiment, there were two choices. We could compare the theoretical and experimental $g(R)$ directly, or we could compare the Fourier transform of our $g(R)$ with the experimental X-ray scattering factor. We have chosen the latter for the following reason. Our $g(R)$ is doubtless somewhat in error because of approximations in its computation, but it provides a smooth, well-defined curve for the Fourier inversion. On the other

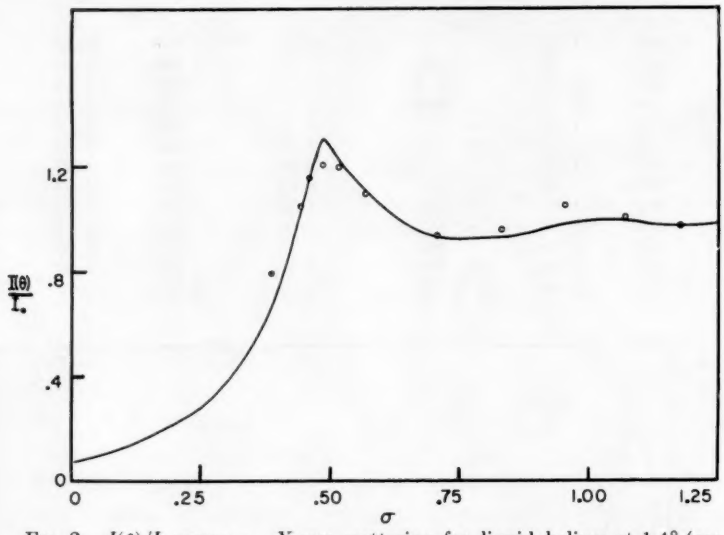

FIG. 2.—$I(\theta)/I_0$ versus σ. X-ray scattering for liquid helium at 1.4° (see Gordon, Shaw, and Daunt, *Phys. Rev*, **96**, 1444, 1954) versus $\sigma = a/\lambda \sin(\theta/2)$, with $a = 2.99$ Å. The solid curve gives the experimental values of Gordon, Shaw, and Daunt. The circles are the theoretical values.

hand, inversion of $I(\theta)/I_0$ to give $g(R)$ involves drawing a smooth curve through points which are subject to random experimental error and, in addition, involves $I(\theta)$ for both small and large angles, which is experimentally difficult to determine. Our $g(R)$ and that determined by Reekie and Hutchinson[7] do, indeed, show discrepancies aside from the scale effect mentioned above. How much of this is to be attributed to the action of the Fourier inversion on experimental error and how much to our approximations is not at all apparent.

In conclusion, we wish to thank Drs. W. L. Gordon, J. G. Daunt and C. H. Shaw, of the Ohio State University, for supplying us with their new X-ray scattering data in advance of publication. We also wish to thank Dr. Marshall Rosenbluth, of the Los Alamos Scientific Laboratory, for providing us with his preliminary estimates of the zero-point energy of a Bose fluid of rigid spheres, calculated by the Monte Carlo technique. At the density of liquid helium at absolute zero, a fluid of rigid spheres possessing a diameter equal to the Lennard-Jones parameter a has an estimated zero-point temperature of 23°, which is higher than our estimate of 12.8° for the actual liquid helium.

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*MATERNAL AGING AND SOMATIC CROSSING OVER OF ATTACHED-X CHROMOSOMES**

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One of the most fundamental problems in genetics is the manner in which crossing over occurs. Although various hypotheses have been advanced from time to time, adequate methods for their experimental testing have usually not been forthcoming. The recent papers of Schwartz[1, 2] are unique in their presentation of specially designed cytogenetic methods for testing his ideas on crossing over. According to his recent scheme,[2] crossing over occurs in two phases: (1) exchanges occur between the two new chromatids of the paired homologous chromosomes, but the remaining two chromatids may become involved in the process through (2) crossing over between the "old" and "new" or "sister" strands of each homologue some time within the same cell division.

In his definitive demonstration of the occurrence of somatic crossing over in *Drosophila melanogaster*, Stern[3] showed that, like meiotic crossing over, this process also occurs in the four-strand stage. It was therefore possible for Schwartz[2] to outline a highly ingenious test of his hypothesis, utilizing somatic crossing over between two arms of an attached-X chromosome, one arm of which carried *y* (yellow body and bristles), the other *sn*^{3} (singed bristles). It may readily be shown that crossing over between an "old" strand of one arm and a "new" strand of the other can lead to the production of two daughter cells, one of which is homozygous *y*, the other homozygous *sn*^{3}.[4] If both contribute sufficiently in the formation of adult tissue, a yellow-singed twin spot will result; if only one of the daughter cells contributes, the resultant mosaic spot will be either yellow or singed. For these particular factors the smallest detectable mosaic spot would be a single bristle, either yellow or singed; the smallest twin spot, two bristles, one yellow, one singed.

If, however, crossing over occurs not at random between either of the two chromatids of one homologue and either of the two chromatids of the other, but in accordance with the Schwartz[2] model, then neither phase 1 nor phase 2 alone will yield daughter cells homozygous for *y* or *sn*^{3}. Both phases must occur in the same

cell division to produce homozygous attached-X chromosomes. Schwartz[2] assumed that (1) sister-strand crossing over is normally rare during development in *D. melanogaster*, and (2) the frequency of sister-strand crossing over may be appreciably increased in the progeny by aging the mother prior to mating. Schwartz therefore performed the experiment of aging attached-X females prior to mating and obtained a ninefold increase in the incidence of twin spots among their progenies. This set of circumstances is so unique that the experimental results appeared to provide a conclusive demonstration of the Schwartz model.

Because the present report concerns our failure to confirm Schwartz's conclusions, as well as the reasons why his results were not to be expected, it seems worth while at this point to examine the two assumptions cited in the preceding paragraph. The second assumption is based on the finding of Brown and Hannah[5] that aging virgin females increases markedly the frequency of eliminations of a ring-X chromosome introduced on subsequent fertilization. The more frequent losses of the ring chromosome may be explained as a result of increased sister-strand crossing over, but other explanations cannot be ruled out on the basis of available evidence. The first assumption is based on the usual failure of most workers to detect much mosaicism in flies carrying a ring-X chromosome. If sister-strand crossing over occurs, ring chromosomes will be transformed into dicentric bridges and be either lost or altered genetically. However, Stern[3] did report a rather high incidence of mosaicism with a ring-X chromosome, and Brown and Hannah[5] also found small mosaic spots in large numbers. Thus, according to the literature cited by Schwartz,[2] his assumptions are certainly not well founded. Further information on the frequency of mosaicism with a ring-X chromosome will be provided below.

Preliminary Observations.—The information to be presented in this section has been on hand for some time, but, because the results were negative and the methods used not so refined as those employed later, it has not been previously published. However, these data did seem sufficiently reliable to indicate that there was little or no effect of aging the maternal parent on the later stages of development in her progeny at the time when somatic crossing over occurs. On the basis of these results, Schwartz's observations seemed quite unexpected and the work in need of repetition.

As may be seen from the first two rows of Table 1, aging has little or no effect on the incidence of somatic crossing over between two rod chromosomes. This result has also been reported by Schwartz[2] and is to be expected either on the basis of his hypothesis or on the hypothesis of a random recombination of nonsister strands. The second two rows, concerning a ring-X chromosome, show that aging increases mosaicism occurring early in development, with the consequent production of large spots, but that the effect decreases as development proceeds. The small spots, of one and two bristles, are very numerous, total about 500 per cent, and demonstrate that the ring-X chromosome is highly unstable during later developmental stages.

The rod chromosomes used in this study were selected for comparison with the ring-X series. With *y* and *sn*^{3} both on the same rod chromosome, only one of the products of a somatic exchange will yield distinguishable tissue in the adult fly, while with *y* and *sn*^{3} each on a different homologue, both products will be distin-

guishable in the twin spots (the infrequent double crossovers will not be considered here). Thus the twin spots are expected to fall in a series of size classes beginning with a minimum of two bristles and ending with a maximum twice as large as the largest spots produced with both *y* and *sn*^{3} on the same chromosome. These expected size classes for the twin spots are indicated by the braces in Table 1. In the last two rows a comparison of the incidence of ring-X mosaicism of aged and unaged cultures will provide a criterion of the effect of aging at the developmental stage at which twin spots are expected to be produced. As shown by the statistical comparison in Table 1, the effect of aging on ring chromosome elimination is not statistically significant during the specified developmental stages. Although the aged series shows more mosaicism in these size classes, the increase is of the order of 40 per cent, which, if not simply attributable to sampling error, may be contrasted with the 900 per cent increase reported by Schwartz.[2]

Materials and Methods.—The methods used in the collection of the data to be offered in the next section have been developed, for the most part, by Dr. Aloha Hannah and represent the most refined techniques yet employed for the study of small mosaics. The maternal parents were kept as virgins either two days (unaged series) or ten to twelve days (aged series) prior to mating, while the fathers were uniformly two days old. After emergence

TABLE 1

COMPARISON OF THE EFFECTS OF AGING OF THE MATERNAL PARENT PRIOR TO MATING ON THE INCIDENCE OF MOSAICISM WITH ROD AND RING CHROMOSOMES

GENOTYPE[a]	MATERNAL TREATMENT[b]	NO. OF ABDOMENS	PER CENT MOSAICISM[c]	NO. OF SPOTS PER SIZE CLASS (SIZE CLASSES EXPRESSED AS BRISTLES PER SPOT)								SIZE-CLASS INDEX[d]
				1	2–3	4–7	8–15	16–31	32–63	64–127	128–255	256–Total
y ac sn^3/ + + + (rod/rod)	Unaged	96	60.4	34	17	7	...	...	...	...	...	... 1.66
	Aged	114	57.0	37	23	5	...	...	...	...	...	... 1.58
y ac sn^3/ + + + (rod/ring, X^{ca})	Unaged	96	481.2	410	44	4	2	2	0	0	0	0 1.29
	Aged	106	536.8	476	56	13	7	4	6	2	0	5 4.17

Braces designate expected size classes of twin spots with *y* +/+ *sn*^{3} (rod/rod) genotype. Statistical comparison of the braced classes for the rod/ring heterozygotes:

Unaged: 50 spots/ 96 abdomens
Aged: 76 spots/106 abdomens
$\chi^2 = 2.82$; df = 1; $P = 0.05$–0.10

[a] The maternal contribution to the genotype is listed first.
[b] The unaged females were kept two days prior to mating; the aged, ten to twelve days.
[c] Total spots as percentage of total abdomens.
[d] The number of spots per size class multiplied by the minimum number of bristles per size class totaled and divided by the total number of abdomens bearing spots.

the offspring were aged two days to assure an approximately equal darkening of the bristles: aging after eclosion will convert some of the phenotypically yellow mosaic spots to the phenotypic wild type[6] and therefore might convert a yellow and singed twin spot into a simple spot consisting only of singed bristles. The two-day-old offspring were then frozen at $-18°$ C.[7] and kept at this temperature until further processing. The abdomens were mounted in Euparal on a microscope slide and then flattened by gentle pressure on the cover slip.[6] A research microscope, with transmitted light and a magnification of 125 or 150×, was used for inspection of the abdomens for mosaics.[6] All mosaic spots were charted on mimeographed abdomen maps, one map being used for each abdomen with any mosaicism. The mapping was carried out by research assistants who had not been apprised of any expected outcome and whose work was easily checked from the permanent abdomen mounts and the permanent record.

Two sets of experiments were carried out separately by each of the present authors. Welshons used an attached-X derived from a Muller-doubler stock ($y\ w^a$. $Dp(1)B^S$) and followed the customary culture methods of the zoölogy department. Brown obtained an attached-X from a triploid stock and followed the genetics department routine. Thanks are due Professor Melvin M. Green for supplying both original stocks.

Single female cultures were employed exclusively. No offspring were used from a culture unless the appearance of yellow and of singed individuals showed the mother to be of the $y + . + sn^3$-genotype. Of the wild-type offspring chosen for inspection, only about 70 per cent will be of the proper genotype, which is that of the mother; since the correction factor for percentage twin spots would be about the same for both the aged and the unaged series, it has not been used in either.

Experimental Results.—The data on frequency of mosaicism in the four sets of experiments are summarized in Table 2. In two sets the flies were divided according to autosomal genotype, to make a total of six pairs of series. As may be readily seen, marked differences in frequency exist among the various series. The last three columns give the results of heterogeneity tests among the individual cultures and indicate that the heterogeneity for the twin spots is probably a reflection of heterogeneity for all spots, or simply for the total amount of somatic crossing-over; when twin spots are considered with respect to all spots (last column), the heterogeneity is significant in only one case. Taken as a whole, the data in Table 2 show that the frequency of somatic crossing over is subject to wide fluctuation because of unknown environmental and genetic factors which are otherwise minor in effect. When the results are considered as a whole (see also total percentages, Table 3), there is no evidence for an influence of maternal aging on the frequency of somatic crossing-over. However, the heterogeneity among the series could have led to any of the three possible conclusions if only one or two pairs of series had been considered (see summary, bottom of Table 2). The frequencies of twin spots reported by Schwartz, 0.77 and 6.77 per cent for the unaged and aged series, respectively, are at the lower range of those cited here. If Schwartz were working with material at the lower range of frequency for twin spots, minor differences in genotype and environment might be expected to produce an exaggerated effect. On the other hand, it is possible to pick out series from Table 2, such as 2*b* M and

TABLE 2

EFFECT OF AGING MATERNAL PARENT ON THE INCIDENCE OF SOMATIC MOSAICISM IN FLIES BEARING AN ATTACHED-X CHROMOSOME WITH $y +$ AND $+ sn^3$ ON OPPOSITE ARMS

Series No.[a]	Autosomal Genotype[b]	Maternal Treatment[c]	Total Cultures	Total Abdomens	Total Mosaics: All Spots (Per Cent)	Total Mosaics: Twin Spots (Per Cent)	Heterogeneity χ^2: All Spots/Abdomens[d]	Heterogeneity χ^2: Twin Spots/Abdomens[d]	Heterogeneity χ^2: Twin Spots/All Spots[d]
SST 1*a*	+	Unaged	4	118	103	19.5	23.6**	8.85*	4.04
1*b*	+	Aged	1	17	100	11.8	...	...	...
SST 2*a*	M	Unaged	5	73	58.9	4.1	19.2**	7.56	5.81
2*b*	M	Aged	8	46	89.1	4.3	27.5**	9.56	3.95 (6)
SST 2*a*	Cy	Unaged	5	79	87.8	21.1	1.42	5.54	3.99
2*b*	Cy	Aged	9	77	123	40.2	22.7**	16.7*	3.66 (7)
W *a*	+	Unaged	8	139	65.5	10.8	17.0*	4.33	14.2*
b	+	Aged	13	178	68.5	11.8	16.4	9.66	8.23
W *c*	M	Unaged	17	166	293	35.5	43.4**	104.1**	19.2
d	M	Aged	24	107	137	19.6	47.2**	48.0**	26.9 (21)
W *c*	+	Unaged	17	171	83.0	12.9	22.2	19.0	11.2 (15)
d	+	Aged	25	138	60.9	12.3	32.8	20.1	24.8 (19)

SUMMARY OF COMPARISONS

ca.	For All Spots	For Twin Spots
Unaged = Aged	2	3
Unaged > Aged	2	2
Unaged < Aged	2	1

[a] Flies of the same series designation but of different genotype were siblings within the individual cultures.
[b] M: *al b M(2)S7/+*; Cy: *In (2R)Cy al L[4] sp[2]/+*.
[c] The unaged females were kept two days prior to mating; the aged, ten to twelve days.
[d] Significant χ^2 values for heterogeneity among cultures are starred: *, $p < 0.05$; **, $p < 0.01$; degrees of freedom are one less than the number of cultures, except for some items in the last column where they are given in parentheses.

TABLE 3

THE SIZE-CLASS DISTRIBUTIONS OF THE ATTACHED-X MOSAICS

TOTAL SIZE-CLASS DISTRIBUTIONS

	Total Abdomens	Size Classes (Bristles per Spot): 1	2–3	4–7	8–16	17–32	Total Mosaics	Per Cent Mosaics
Nontwin spots:								
Unaged series	757	600	194	27	2	...	823	109
Aged series	563	276	104	31	1	...	412	73
		$\chi^2 = 12.4$; df $= 2^a$; $p < 0.01$						
Twin spots:								
Unaged series	757	...	89	46	6	0	141	18.6
Aged series	563	...	43	40	10	1	94	16.7
		$\chi^2 = 8.87$; df $= 2^a$; $p < 0.02$						

SIZE-CLASS INDEXES[b]

Series	Maternal Treatment	Indexes of Nontwin Spots	Indexes of Twin Spots
SST 1*a*	Unaged	1.46	[c]
1*b*	Aged	1.60	[c]
SST 2*a* M	Unaged	1.35	[c]
2*b* M	Aged	1.46	[c]
SST 2*a* Cy	Unaged	1.38	3.58
2*b* Cy	Aged	1.53	5.07
W *a*	Unaged	1.39	2.53
b	Aged	1.68	3.56
W *c* M	Unaged	1.34	2.44
d M	Aged	1.35	2.29
W *c*	Unaged	1.24	3.00
d	Aged	1.45	3.41

[a] The two classes with the smallest numbers were combined.
[b] See footnote[d], Table 1, for calculation of indexes.
[c] These series had too few twin spots to warrant comparison; see Table 2.

2*b* Cy, which also show a nine- or tenfold difference. The two series just cited, as well as the two 2*a*, W *c*, and W *d* series, were from sibs emerging in the same individual cultures but classified according to autosomal genotype; thus the differences between them in frequency of mosaicism most likely stem from genetic rather than from environmental differences. Inspection of frequencies for the individual cultures, which cannot be cited here in detail, strengthens the impression that the variation in frequency in this experiment is largely under genetic control.

The data were searched for evidence of other effects of aging, and Table 3 presents the information on size-class distribution. Although the total frequency of spots, both twin and nontwin, was lower in the aged than in the unaged series, the spots in the aged series were somewhat larger. In contrast to the heterogeneity in frequency, the consistency of the size increase is noteworthy. In the two instances in which differences occurred in the opposite direction (top two rows, Table 1; series W *c* M and W *d* M, twin spots, Table 3), the magnitude of the difference was small. It is further of interest to note that the increase in average size is relatively slight, quite in keeping with the nonsignificant difference shown in the ring-X comparison for these developmental stages (Table 1). The total size-class distribution of the twin spots (Table 3) proved to be exactly that predicted from the size-class distribution of the single spots reported in Table 1. Thus the size-class distribution seems to be a much more consistent feature of somatic crossing-over than frequency and may therefore prove a more reliable indicator of environmental and genetic effects.

An unexpected result was the much greater incidence of mosaicism in flies heterozygous for the Curly inversion than in their siblings heterozygous for a Minute (series SST 2*a* and 2*b*, Table 2). That heterozygous inversions increase the frequency of meiotic crossing-over in other chromosomes is well substantiated. A detailed study of the influence of inversions on somatic crossing over might provide further information of value.

Discussion.—The results of the preliminary experiment with rod and ring chromosomes show that an effect of aging on somatic crossing over in attached-X chromosomes is an unlikely expectation, while the experiments with the attached-X chromosomes confirm this prediction. The present work thus offers no evidence in favor of Schwartz's two-phase model of crossing over, nor does it in any way disprove his hypothesis. Like so many ideas on the mechanism of crossing over, this one is also left without experimental foundation.

Schwartz's two-phase model of crossing over must be carefully distinguished from sister-strand crossing over itself, evidence for which has recently been obtained by Schwartz[2] in a study of a long ring chromosome in *Zea mays*. The stability of the ring-X chromosome in *Drosophila* has been considered to be at variance from the observations of marked instability of ring chromosomes in plants.[8] Stern's[3] results, as well as those reported here, show that the ring-X chromosome is highly subject to change or loss in late developmental stages. Further experiments are in progress in an attempt to obtain more information on the importance of sister-strand and nonsister-strand crossing over with the ring-X chromosome during these developmental periods.

Summary.—In a recent publication Schwartz reported that aging of attached-X females prior to mating resulted in a ninefold increase in the frequency of twin spots

in the progenies. This result was offered in support of a two-phase model of crossing-over.

Preliminary observations of the present authors had shown that aging had little or no effect at the late developmental stages at which somatic crossing over occurs, and, in an extensive series of tests, Schwartz's experimental findings were not confirmed. These results do not disprove the two-phase hypothesis but show that the method chosen to provide supportive evidence will not do so.

The size-class distribution of mosaic spots seems to be one of the most consistent features of somatic crossing-over. Aging may somewhat increase the size of the spots.

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[1] D. Schwartz, *Genetics*, **38**, 251–260, 1953.

[2] D. Schwartz, *ibid.*, **39**, 692–700, 1954. This paper should be consulted for the relationship of Schwartz's to Belling's ideas on crossing-over.

[3] C. Stern, *Genetics*, **21**, 625–730, 1936.

[4] The diagrams by Schwartz (see n. 2) help materially in visualizing the various types of crossovers and their products.

[5] S. W. Brown and Aloha Hannah, these PROCEEDINGS, **38**, 687–693, 1952.

[6] Aloha Hannah, *J. Exptl. Zool.*, **123**, 523–560, 1953.

[7] The freezing method was adopted at the suggestion of Mr. George Brosseau and was checked by tests of yellow, singed, and wild-type flies as well as various sorts of gynandromorphs. There was no evidence that freezing altered the appearance of the bristles in any way.

[8] In *Zea mays*, Barbara McClintock, *Genetics*, **23**, 315–376, 1938; in *Nicotiana tabacum*, K. R. Stino, *J. Heredity*, **31**, 19–24, 1940.

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*ABERRANT RECOMBINATION OF PYRIDOXINE MUTANTS OF NEUROSPORA**

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Giles[1] has reported that crosses between certain inositol mutants of *Neurospora* give infrequent wild recombinants, and from the segregation of linked markers it appeared that multiple crossovers might sometimes be involved. The present author subsequently observed that crosses between two closely linked or allelic pyrimidine mutants, one of which carried a "colonial" marker, gave occasional genetically wild offspring, but roughly half of these carried the "colonial" marker and half did not. This was surprising, since the marker appeared to be located only two or three units from the pyrimidine locus or loci, and, if the wild recombinants arose from crossovers between the two pyrimidine mutants, all would be expected to be alike with respect to the marker unless, in half the cases, a second crossover

occurred between the pyrimidine locus and the locus of "colonial." These observations will be described in detail elsewhere. Unfortunately, a closely linked marker on the other side of pyrimidine is not known, and, moreover, crosses between these mutants are regularly semisterile, so that it is not possible to observe complete asci. However, another pair of mutants, requiring pyridoxine, also closely linked to "colonial," was found by Dr. T. H. Pittenger (personal communication) to exhibit the same behavior, and these mutants give fertile crosses. When a suitable marker was located[2] on the other side of pyridoxine, a study of recombinants among random spores and in complete asci was undertaken.

Recombinants among Random Spores.—The markers used are *co* (70007), "colonial," and *pyr 1* (263), pyrimidine. The pyridoxine mutants are *pdxp* (39106), pH-sensitive,[3] and *pdx* (37803), not pH-sensitive. Data showing the location of the markers with respect to *pdx* have been presented.[2] The "colonial" marker is located about 10–13 units from the centromere in linkage group D; *pyr 1*, about 4–5 units nearer the centromere; and *pdx*, between these two, 0.5–1 unit from *pyr 1*. Data from more recent crosses are consistent with the location of *pdxp* at approximately the same position as *pdx*. The double mutant *pyr 1 pdx* was obtained from an ascus from the cross *pyr 1* + *co* × + *pdx* +, and *pdxp co* was found among random spores from *pyr 1* + *co* × + *pdxp* + by isolating spores producing mycelium which grew on minimal plus pyridoxine and showed the "colonial" growth habit. These were then tested for pyridoxine requirement.

Germinated spores from the crosses *pdx* × *pdxp*, *pdx* × *pdx*, and *pdxp* × *pdxp* were counted on minimal agar supplemented with pyrimidine. The results are given in the accompanying tabulation. Pittenger[4] has shown that pseudo-wilds

	pdx or *pdxp*	+ *co*	+ +
+ *pdxp co* R 1 a × *pyr pdx* + R 1 A	7,255	6	8
+ *pdxp co* R 1 a × *pyr pdx* + R 7 A	14,278	15	15
+ *pdx co* R 2 a × + *pdx* + R 4 A	13,554	0	0
+ *pdx co* R 2 a × *pyr pdx* + R 7 A	8,271	0	0
+ *pdxp co* R 1 a × + *pdxp* + R 1 A	22,747	0	0

are found from *pdx* × *pdxp* crosses, and they were also observed here. With practice, however, these can be distinguished from true wilds by their growth habit on the agar plates, and they are not included in the counts given. Of the forty-four not *pdx* or *pdxp* progeny counted, twelve + *co* and twenty + + were isolated and tested for pyrimidine requirement. Among these were the following types: five + + *co*, seven *pyr* + *co*, seven + + +, and thirteen *pyr* + +. Two of each of the four types were crossed to wild-type protoperithecia. Random spores from these crosses were of the types expected if the recombinants have the genotypes corresponding to their phenotypes.

The distance between *pyr* and *co* in the heterozygous crosses, checked by plating spores on minimal plus pyridoxine, was about 4–5 units. Counts were as follows:

	pyr +, + *co*, *pyr co*	+ +
+ *pdxp co* × *pyr pdx* + R 1	3,844	85
+ *pdxp co* × *pyr pdx* + R 7	2,330	64

Recombinants among Segregants in Asci.—Dissections of asci from the two heterozygous crosses were done on plates of minimal agar supplemented with pyrimidine.

The spore pairs were separated, so that after germination they could be transferred to slants, and the plates were then heat-treated and incubated at 25° C. After about 17 hours the plates were examined under a microscope, and the spore pairs of asci which contained recombinants were transferred to slants of complete medium. The cultures thus obtained were tested on media suitable for the classification of all segregants. Among 988 asci dissected, 585 gave complete germination (at least one member of each spore pair germinated), and among these were found four showing recombinants. The segregants contained in these were as shown in the accom-

	Spore Pairs			
Ascus	1	2	3	4
1	+ *pdxp co* A	+ + *co* A	*pyr pdxp* + a	*pyr pdx* + a
2	*pyr pdx* + a	*pyr pdx* + a	+ + *co* A	+ *pdxp co* A
3	+ + + a	+ *pdxp co* a	*pyr pdx* + A	*pyr* + *co* A
4	+ *pdxp co*	+ + *co*	*pyr pdx* +	*pyr pdx* +

panying tabulation. It will immediately be seen that if the double mutant *pdx pdxp* is present in these asci it must have the phenotype of *pdx* in asci 2 and 4, but of *pdxp* in ascus 1, and in ascus 3 its phenotype must be that of *pdxp* in spore pair 2 and that of *pdx* in spore pair 3. Although, in the author's experience with *Neurospora* mutants, no double mutant has been observed to exhibit such behavior, backcrosses to the two parent types were made in order to test for the presence of the double mutant. Since the crosses *pdx* × *pdx* and *pdxp* × *pdxp* failed to give recombinants, it would be expected that *pdx pdxp* would not give recombinants when crossed either to *pdx* or to *pdxp*. Random spores from crosses of all *pdx* and *pdxp* segregants from asci 1, 2, and 3 to *pdx* and to *pdxp* were plated on minimal plus pyrimidine and examined. Recombinants were found among spores from all *pdx* × *pdxp* crosses but not among those from *pdx* × *pdx* and *pdxp* × *pdxp* crosses. Spore counts are given in Table 1. It appears, then, that these asci did not contain a conventional double mutant such as would be expected to arise from a crossover between *pdx* and *pdxp* if they represent conventional independent loci.

The not *pdx* or *pdxp* segregants from the same three asci, when crossed to wild protoperithecia, gave, as far as could be seen, no *pdx* or *pdxp* progeny but only such progeny as would be expected from + + +, + + *co* and *pyr* + *co* strains.

Dry weights of mycelium were obtained from 125-ml. flask cultures of all segregants from asci 1, 2, and 3 and of nine recombinants from random spores—one + + +, four + + *co*, one *pyr* + *co*, and three *pyr* + +. The media used were minimal at pH 5.1, with and without a supplement of pyridoxine, and minimal at pH 7.1. All flasks were supplemented with pyrimidine. The *pdx* and *pdxp* isolates showed no significant differences from the parent strains. The not *pdx* or *pdxp* recombinants gave no response to pyridoxine.

Since most of the asci not showing recombinants were not transferred from the plates, classifications of *pdx*, *pdxp*, and *pyr* could not usually be made, but, if the plates were examined after 40 hours, *co* could be scored, due to the slow growth on minimal agar of *pdx* and *pdxp*. There were 123 second-division segregations in 584 asci, giving a centromere distance of 11 units. In some cases, at least, asci containing + *pdxp* + could be detected, since *pdx* grows more slowly on minimal agar than *pdxp*, and when *co* is removed from *pdxp* this is more obvious. Of six asci isolated and tested, five were of the following constitution:

+ *pdxp* + + *pdxp co* *pyr pdx co* *pyr pdx* +,

indicating a crossover between *pdxp* and *co*. One was of the constitution

$$+\ pdxp\ co \qquad pyr\ pdx\ co \qquad +\ pdxp\ + \qquad pyr\ pdx\ +,$$

indicating a crossover between the centromere and *pyr* as well. These asci do not give a measure of the frequency of crossing-over in the region *pdxp* to *co*, since not all asci which appeared to contain + *pdxp* + were tested; but they serve to show that these crosses give apparently normal asci of the constitution expected to result from such a crossover.

TABLE 1

BACKCROSSES OF *pdx* AND *pdxp* SEGREGANTS FROM ASCI 1, 2, AND 3

									pdx or *pdxp*	+ *co*	+ +
+	*pdx*	+	a	×	+	*pdxp*	*co*	1-1 A	6,437	5	3
+	*pdxp*	*co*	a	×	+	*pdxp*	*co*	1-1 A	3,238	0	..
+	*pdx*	*co*	A	×	*pyr*	*pdxp*	+	1-3 a	4,860	2	3
+	*pdxp*	+	A	×	*pyr*	*pdxp*	+	1-3 a	4,175	..	0
+	*pdxp*	+	A	×	*pyr*	*pdx*	+	1-4 a	7,036	..	10
+	*pdx*	*co*	A	×	*pyr*	*pdx*	+	1-4 a	4,106	0	0
+	*pdxp*	+	A	×	*pyr*	*pdx*	+	2-1 a	2,276	..	3
+	*pdx*	*co*	A	×	*pyr*	*pdx*	+	2-1 a	5,269	0	0
+	*pdxp*	+	A	×	*pyr*	*pdx*	+	2-2 a	3,438	..	5
+	*pdx*	*co*	A	×	*pyr*	*pdx*	+	2-2 a	3,795	0	0
+	*pdx*	+	a	×	+	*pdxp*	*co*	2-4 A	3,234	1	3
+	*pdxp*	*co*	a	×	+	*pdxp*	*co*	2-4 A	3,085	0	..
pyr	*pdx*	+	A	×	+	*pdxp*	*co*	3-2 a	2,240	2	3
+	*pdxp*	+	A	×	+	*pdxp*	*co*	3-2 a	3,836	0	0
+	*pdxp*	*co*	a	×	*pyr*	*pdx*	+	3-3 A	2,588	4	0
+	*pdx*	*co*	a	×	*pyr*	*pdx*	+	3-3 A	4,656	0	0

Discussion.—But for the apparent absence of *pdx pdxp* double mutants in the asci, the frequencies of the four recombinant types among random spores and the segregations in asci could, perhaps, be explained by assuming that *pdx* and *pdxp* are independent loci and that in the region *pyr* to *co* multiple crossovers occur as frequently as single crossovers. There is, however, the fact that data[2] from other crosses involving this region indicate a much higher frequency of single crossovers, and, to the author's knowledge, multiple crossovers have not been found in asci. It therefore seems desirable to attempt to account for the recombinants on the basis of some mechanism other than crossing-over.

The phenomenon usually referred to as back mutation hardly seems to furnish a suitable explanation because of the stability of *pdx* and *pdxp* in vegetative culture and the failure to find recombinants among spores from *pdx* × *pdx* and *pdxp* × *pdxp* crosses, in contrast to the rather high frequency of recombinants among spores from *pdx* × *pdxp* crosses. Also, in ascus 1, it would be necessary to assume a two-strand double crossover between *pyr* and *co*, as well as back mutation of *pdx*, in order to account for the segregants observed.

In order to explain abnormal segregations which cannot be accounted for by polyploidy or chromosome loss in yeast asci, Lindegren[5] has revived the theory of gene conversion by means of which Winkler[6] once sought to account for all genetic recombination. Conversion of a dominant gene to its recessive allele partly, at least, through the influence of the recessive was suggested to explain asci in which, for example, three spores carried the recessive gene and only one the dominant, although another linked gene pair segregated normally. If it is assumed that *pdx* and *pdxp* are alleles, and in the author's opinion there is no satisfactory evidence that they are not, the formation of not *pdx* or *pdxp* recombinants may be pictured

as resulting from some sort of interaction between the two alleles, perhaps analogous to gene conversion. Since the two mutants are phenotypically different and since they are capable of forming a pyridoxine-independent pseudo-wild, they might be considered to represent different defects at the same locus. Hence, together they may collaborate to obliterate the pyridoxine requirement, not only in a pseudo-wild, but also occasionally at meiosis when the chromosomes are being duplicated. If each mutant carries in the normal condition the part of the locus which is defective in the other and if the chromosome strands are closely associated at the time they are being duplicated, a recombinant might arise as a result of the normal part of one allele being duplicated twice and incorporated into each new strand. This scheme is diagrammed in Figure 1 for each of the three different asci found. If such abnormal duplication actually occurs, segregation ratios of three

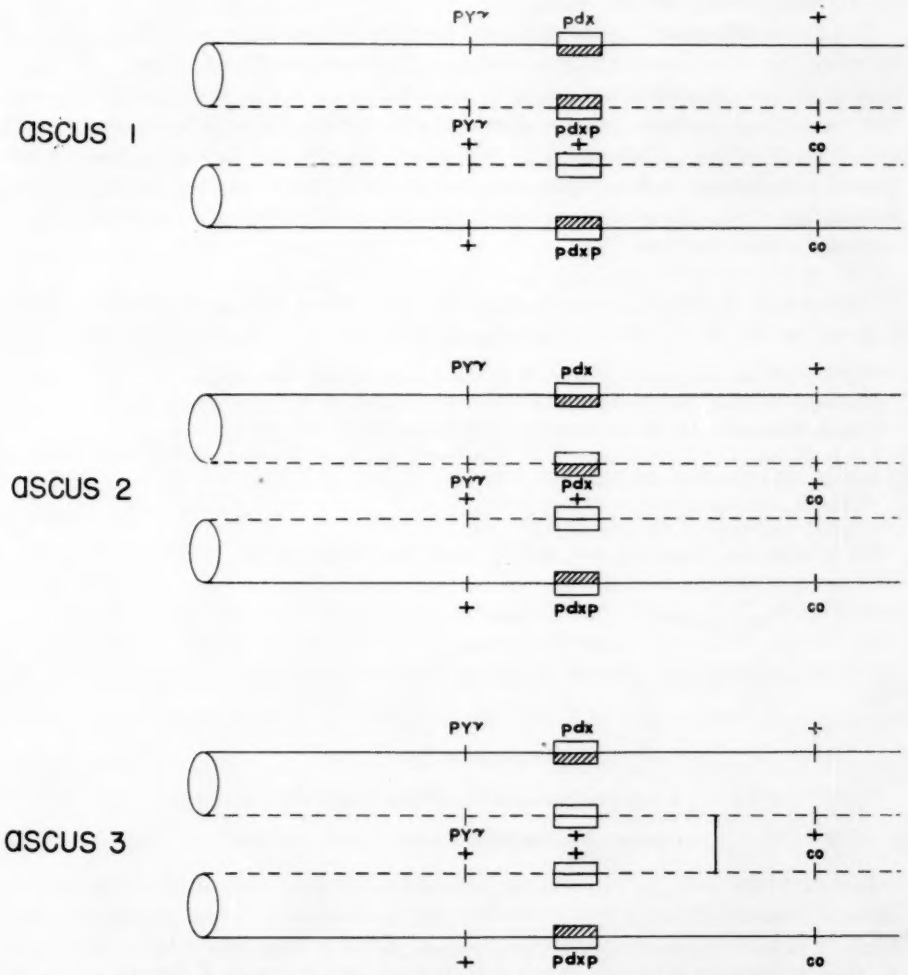

FIG. 1.—Hypothetical scheme to account for origin of not *pdx* or *pdxp* segregants in asci by repeated duplication of one or both normal parts of the locus and failure of corresponding defective parts to be duplicated. Newly formed chromosome strands are represented as broken lines. Parts of the locus assumed to be defective are shaded.

pdx to one *pdxp* and the reverse would be expected to result from repeated duplication of the defective parts of the locus. These might be detected if asci were examined on minimal at pH 7, but this has not been tried. Also, segregants carrying both defective parts of the locus would be expected and might behave as the double mutant in backcrosses, but they might not be distinguishable from *pdx* by any other test.

In order to explain the high frequency of recombinants showing recombination of *pyr* and *co*, it might be supposed that a crossover in the vicinity of the *pdx*, *pdxp* locus brings about a more intimate association of the duplicating strands and thereby increases the probability of abnormal duplication. It might even be supposed that the event which gives rise to crossing-over took place in each case very near the *pdx*, *pdxp* locus but that this event results in an actual exchange in, statistically, only half the cases.

The above scheme is, of course, purely speculative, but it serves to suggest a way in which the recombinants might arise as a result of abnormal meioses. The frequency of such abnormalities might be expected to vary a good deal with different loci, depending, perhaps, on the degree of structural difference between the wild and mutant alleles. This might be one reason for previous failure to observe abnormal segregations in *Neurospora* asci; but another reason may be that such segregations have been dismissed as mistakes because of the absence of suitable markers in many crosses observed.

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*SPLITTING OF VALUATIONS IN EXTENSIONS OF LOCAL DOMAIN II**

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In a previous note[1] Zariski and the author have proved that, if (R, M) is a regular local domain of dim $s > 1$, such that the quotient field K and the residue field R/M of R have equal characteristics, and, if K^* is a finite separable extension of K, then there exist infinitely many real discrete valuations of K having center M in R which split in K^*.[2] The purpose of this note is to generalize this result by relaxing the regularity condition on R and drawing, at the same time, a stronger conclusion. We state this generalization as our

THEOREM. *Let (R, M) be a local domain of dimension $d > 1$, such that either (a) R admits a nucleus,[3] or (b) R is regular and has the same characteristic as its residue field. Let K be the quotient field of R and K^* a finite separable extension of K. Then there exist infinitely many real discrete valuations v of K with the following two properties:* (1) *v has center M in R, and* (2) *any K^*-extension v^* of v has degree 1 over v,[4] or, equivalently, v has exactly $[K^*:K]$ distinct extensions to K^*.*

We shall divide the proof into several lemmas.

LEMMA 1. *Let (R, M) be a local domain with quotient field K and K^* a Galois extension of K. Let o be either the valuation ring of a real valuation of K or a local domain with quotient field K, and let m be the maximal ideal in o. Let $\bar{M}$ and $\bar{m}$ be certain maximal ideals in the integral closures $\bar{R}$ and $\bar{o}$ in K^* of R and o, respectively. Let $R^* = \bar{R}_{\bar{M}}$, $M^* = \bar{M}R^*$, $o^* = \bar{o}_{\bar{m}}$, and $m^* = \bar{m}o^*$. Let G and g be the splitting groups of M^* and m^* over M and m, respectively. Assume that $R \subset o$, $m \cap o = M$, and $R^* \subset o^*$. Then $g \subset G$.*

Proof: We have $m^* \cap \bar{R} = M^* \cap \bar{R}$, and hence, for any element t of g, $t(M^* \cap \bar{R}) = t(m^*) \cap t(\bar{R}) = m^* \cap \bar{R} = M^* \cap \bar{R}$, i.e., $t \in G$.

LEMMA 2. *Let (o, m) be a two-dimensional regular local domain, such that the quotient field K and the residue field o/m of o have the same characteristic. Let K^* be a (finite) Galois extension of K with $K^* \neq K$. Then there exists a two-dimensional regular local domain (R, M) with the following three properties:* (1) *$o \subset R \subset K$.* (2) *$M \cap R = m$.* (3) *If K_s is the splitting field over M of a maximal ideal in the integral closure of R in K^*, then $K_s \neq K$.*

Proof: By Theorem 1 of AZ,[1] we can find a real discrete valuation w of K with center m in o such that w splits in K^*. Let d be the o-dimension of w. Then $d = 0$ or 1.[5] We deal with the two cases separately.

Case 1, $d = 0$. Then, for any integer i, there is a (unique) ith quadratic transform (o_i, m_i) of (o, m) such that w has center m_i in o_i, i.e., *the* quadratic sequence $o = o_0 \subset o_1 \subset o_2 \subset \ldots$ along w starting from o is infinite. By a theorem of Zariski,[6] $\bigcup_{i=1}^{\infty} o_i = R_u$, where u is a valuation of K. Then $R_w \supset R_u$ and $M \subset_w \bigcup_{i=1}^{\infty} m_i = M_u$. Therefore, $R_w = R_u$ and $w = u$. Let $\bar{o}_i$ be the integral closure of o_i in K^*, and let w^* be a K^*-extension of w. Let $p_i = \bar{o}_i \cap M^{w*}$, $o_i^* = (\bar{o}_i)_{p_i}$, $m_i^* = p_i o_i^*$. By Lemma 1, for some integer n we have that the splitting fields of m_i^* over m_i are all equal for $i \geqq n$ and that, if we denote this common splitting field by K_s, then $K_s \supset K_s^*$, where K_s^* is the splitting field of w^* over w. Suppose, if possible, that $K_s \neq K_s^*$. Then w has a K^*-extension $\bar{w}$ different from w^*, such that $\bar{w}$ has center m_i^* in o_i^* for all i.[6] Therefore, $\bigcup_{i=1}^{\infty} o_i^*$ is not a valuation ring, and hence there exists a valuation t^* of K^* such that t^* has center m_i^* in o_i^* and the o_i^*-dimension of t^* is 1 for all i. Let t be the K-restriction of t^*. Then t has center m_i in o_i, and the o_i-dimension of t is 1 for all i. This is a contradiction, since $\bigcap_{i=1}^{\infty} o_i = R_w$. Therefore, $K_s = K_s^*$. Since w has more than one extension to K^*, $K_s^* \neq K$. Now any (o_i, m_i) with $i \geqq m$ will do the job of (R, M).

Case 2, $d = 1$. Suppose, if possible, that the quadratic sequence $o = o_0 \subset o_1 \subset o_2 \subset \ldots$ along w starting from o is infinite. Then, as in case 1, $\bigcup_{i=1}^{\infty} = R_w$. This is a contradiction, since $d > 0$. Therefore, the sequence $o_0 \subset o_1 \subset o_2 \subset \ldots$ is finite. Let

$o_n = \bar{R}$ be its last member, and let $\bar{M}$ be the maximal ideal in $\bar{R}$. Let (x, y) be a minimal basis of $\bar{M}$, $S = \bar{R}[y/x]$, $N = xS$, and $y^* =$ the N-residue of y. Then $S_N = R_w$, and $(\bar{R}/\bar{M})\ (y^*) = R_w/M_w$. Let A be any maximal ideal in S containing y, let t be the valuation of R_w/M_w given by the ideal A/N in $(\bar{R}/\bar{M})\ [y^*]$, and let w_1 be the valuation of K which is composed of w and t. Then w_1 has center m in o, the o-dimension of w_1 is zero, and w_1 splits in K^*. This reduces case 2 to case 1.

LEMMA 3. *The theorem is true in case b, assuming that $d = 2$ and that K^*/K is Galois.*

Proof: We shall apply induction to $[K^*:K]$. The lemma is trivial for $[K^*:K] = 1$. Now let $[K^*:K] = n > 1$, and assume that the lemma is true whenever $[K^*:K] < n$. Let $\bar{M}$ be a maximal ideal in the integral closure $\bar{R}$ of R in K^*. Let $R^* = \bar{R}_M$, $M^* = \bar{M}\,R^*$, and $K_s =$ the splitting field of M^* over M. By Lemma 2, we can assume that $K_s \neq K$. Let $R_s = R^* \cap K_s$ and $M_s = M^* \cap K_s$. By Theorem 2 of AZ, $M_s = MR_s$ and hence R_s is regular. Therefore, by our induction hypothesis, there exist infinitely many real discrete valuations v_s of K_s with center M_s in R_s such that, if v^* is an extension of v_s to K^*, then v^* is of degree 1 over v_s. Let v_s, v^* be such a pair, and let v be the K-restriction of v_s. Then v has center M in R, and, by Lemma 1, v_s is of degree 1 over v, and hence v^* is of degree 1 over v. Since K^*/K is Galois, all the K^*-extensions of v are of degree 1 over v. The infinitely many choices of v_s give us infinitely many valuations v of the required type.

LEMMA 4. *The theorem is true in case b.*

Proof: Let $x_1, x_2, \ldots, x_d$ be a minimal basis of M. Let $S = R[x_2/x_1, x_3/x_1, \ldots, x_d/x_1]$, $N = (x_1, x_2)S$, $o = S_N$, and $m = No$. Then (o, m) is a two-dimensional regular domain, $o \supset R$, and $m \cap R = M$. Now Lemma 4 follows by applying Lemma 3 to (o, m) and by passing to a Galois extension of K containing K^*.

LEMMA 5. *The theorem is true in case a, assuming that R is complete.*

Proof: By a theorem of Cohen, R contains a coefficient field k. Let $x_1, x_2, \ldots, x_d$ be parameters in R. Let $o = k[[x_1, x_2, \ldots, x_d]]$, $m = (x_1, x_2, \ldots, x_d)o$, and $K_1 = k((x_1, x_2, \ldots, x_d))$. Then, by a theorem of Chevalley, R is a finite o-module.[7] Let K_s and K_s^* be the separable algebraic closures of K_1 in K and K^*, respectively. By Lemma 4, there exist infinitely many real discrete valuations v_1 of K_1 having center m in o such that every K_s^*-extension v_s^* of v_1 is of degree 1 over v_1. Let v_1 and v_s^* be such a pair. Let v^* be the K^*-extension of v^*. Let v_s and v be the restrictions of v^* to K_s and K, respectively. Then v_s has degree 1 over v_1, and hence degree of v over $v_1 = [K:K_s] = [K^*:K_s^*] =$ degree of v^* over v_1. Therefore, v^* is of degree 1 over v. The infinitely many choices of v_1 give us infinitely many valuations v of the required type.

LEMMA 6. *The theorem is true in case a, assuming that K^*/K is Galois.*

Proof: Let $(\bar{R}, \bar{M})$ be the completion of (R, M), and let S be the integral closure of R in K. Since R admits a nucleus, $\bar{R}$ has no nonzero nilpotents,[8] and hence S is a finite R-module.[9] Therefore, S is a semilocal ring. Let N be a maximal ideal in S. By replacing R by S_N, we may assume that already R is integrally closed in K. Let M^* be a maximal ideal in the integral closure R^* of R in K^*. Let $\tilde{R} = R^*_{M^*}$ and $\tilde{M} = M^*\tilde{R}$. Let K_s be the splitting field of $\tilde{M}$ over M, $R_s = K_s \cap \tilde{R}$, and $M_s = K_s \cap \tilde{M}$. By Lemma 1, any real discrete valuation of K_s with

center M_s in R_s has degree 1 over its K-restriction. Hence, by replacing R by R_s, we may assume that M^* is the only maximal ideal in R^*.

Let $(\bar{R}^*, \bar{M}^*)$ be the completion of R^*. We may assume that $\bar{R} \subset \bar{R}^*$[10] and that $\Sigma \subset \Sigma^*$, where Σ and Σ^* are the respective quotient fields of $\bar{R}$ and $\bar{R}^*$.[11] Since K^*/K is separable, Σ^*/Σ is also separable.[10] By Lemma 5, there exist infinitely many real discrete valuations $\bar{v}$ of Σ with center $\bar{M}$ in $\bar{R}$ such that any Σ^*-extension $\bar{v}^*$ of $\bar{v}$ has degree 1 over $\bar{v}$. Let $\bar{v}, \bar{v}^*$ be such a pair, and let v, v^* be the restrictions of $\bar{v}, \bar{v}^*$ to K, K^*, respectively. Then v and $\bar{v}$ have the same residue fields and value groups, and so also v^* and $\bar{v}^*$.[12] Therefore, v^* has degree 1 over v. Since distinct valuations $\bar{v}$ have distinct K-restrictions,[1] the infinitely many choices of $\bar{v}$ give us infinitely many valuations v of the required type.

LEMMA 7. *The theorem is true in case a.*

Proof: Apply Lemma 6 to a Galois extension of K containing K^*.

This completes the proof of our theorem.

It was conjectured by Krull and proved by Nagata that, if (R, M) is a local domain integrally closed in its quotient field K, K^* a Galois extension of K, M^* a maximal ideal in the integral closure R^* of R in K^*, K_s the splitting field of M^* over M, $R_s = (R^* \cap K_s)_{M^* \cap K_s}$, and $M_s = (M^* \cap K_s)R_s$, then $M_s = MR_s$. Using Theorem 1 of AZ, in AZ we gave a new proof of the Krull-Nagata theorem, assuming that R admits a nucleus and is regular. In a similar fashion, the theorem of the present note gives us a proof of the Krull-Nagata theorem for arbitrary local domains R admitting a nucleus.

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[1] S. Abhyankar and O. Zariski, "Splitting of Valuations in Extensions of Local Domains," these PROCEEDINGS, **41**, 84–90, 1955. We shall refer to this note as "AZ" and shall use the notations introduced there.

[2] Theorem 1 of AZ.

[3] P. Samuel, *Algèbre locâle* (Paris, 1953).

[4] By the degree of v^* over v we mean the product of the ramification index and the residue degree of v^* over v.

[5] O. Zariski, "A Simple Analytical Proof of a Fundamental Property of Birational Correspondences," these PROCEEDINGS, **35**, 62–66, 1949.

[6] O. Zariski, "Applicazioni geometriche della teoria delle valuazioni," *Rend. mat.* (Rome), Ser. V, **13**, 1–38, 1954. See the theorem on p. 25 and the considerations on pp. 27–28. Although Zariski assumes that the local domains under consideration are quotient rings on algebraic varieties, his proofs are valid for arbitrary local domains.

[7] C. Chevalley, "On the Theory of Local Rings," *Ann. Math.*, **44**, 690–708, 1943. See Proposition 3.5. The infiniteness of the basic field is not used in the proof of this proposition.

[8] C. Chevalley, "Intersections of Algebraic and Algebroid Varieties," *Trans. Am. Math. Soc.*, **57**, 1–85, 1945. See Lemma 1.9.

[9] M. Nagata, "Some Remarks on Local Rings. II," *Mem. Coll. Sci. Univ. Kyoto*, Ser. A, **28**, Math. No. 2, 109–120, 1953. See Corollary 2 of the Appendix.

[10] Chevalley, *Ann. Math.*, **44**, 690–708, 1943, Proposition 2.7. Let $x_1, x_2, \ldots, x_h$ be an R-basis of R^*. Then $x_1, x_2, \ldots, x_h$ is an $\bar{R}$-basis of $\bar{R}^*$. Since the x_i are separable algebraic over K and hence over Σ, $\Sigma^* = \Sigma(x_1, x_2, \ldots, x_h)$ and Σ^*/Σ is separable.

[11] By a theorem of Zariski, $\bar{R}$ and $\bar{R}^*$ are domains; see Samuel, *op. cit.*

[12] This follows by the argument used in the proof of Lemma 4 of AZ.

ON THE EXISTENCE OF THE DERIVATIVE OF MARKOFF TRANSITION PROBABILITY FUNCTIONS

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We consider a matrix of functions $p_{ij}(t)$ $(i, j = 1, 2, \ldots)$, $0 \leqq t, t_1, t_2 < \infty$, such that

$$\text{I.} \quad p_{ij}(t) \leqq 0,$$

$$\text{II.} \quad \sum_j p_{ij}(t) = 1,$$

$$\text{III.} \quad p_{ij}(t_1 + t_2) = \sum_k p_{ik}(t_1)p_{kj}(t_2),$$

$$\text{IV.} \quad \lim_{t \to 0} p_{ij}(t) = \delta_{ij}.$$

It has been shown by Doob[1] that the $p_{ij}(t)$ are continuous and have derivatives at 0. The problem of finding conditions under which the functions have derivatives at all points of $[0, \infty]$ has been brought up in several places. It is suggested, for example, by Kolmogoroff.[2] Recently it has been shown by Juskewicz, in a paper not yet published, that, in general, the functions are not differentiable at all points of $[0, \infty]$. In this paper it is shown that, *if the derivative of* $p_{ii}(t)$ *at* $t = 0$ *is finite, then the* $p_{ij}(t)$ $(j = 1, 2, \ldots)$ *have continuous derivatives on* $[0, \infty]$.

We adopt the following notation: $Dp_{ij}(0) = q_{ij}$, $j \neq i$; $Dp_{ii}(0) = -q_i$; $\Delta_{ij}(t_1, t_2) = (p_{ij}(t_2) - p_{ij}(t_1))/(t_2 - t_1)$; using I, II, III, and IV, we see that for h sufficiently small

$$-q_i - \epsilon < \Delta_{ii}(0, h)p_{ij}(t) \leqq \sum_k \Delta_{ik}(0, h)p_{kj}(t) = \Delta_{ij}(t, t + h) \leqq \sum_{k \neq i} \Delta_{ik}(0, h)p_{kj}(t) < q_i + \epsilon. \quad (1)$$

Since, by a result of Dini, the derivatives and the difference quotient of a continuous function have the same upper and lower bounds on any interval, we conclude that $|\Delta_{ij}(t_1, t_2)| \leqq q_i$ for $j = 1, 2, \ldots, 0 \leqq t_1, t_2$. We now consider the functions on a finite interval $0 \leqq t \leqq \bar{t}$; since a monotone sequence of continuous functions with a continuous limit converges uniformly on any closed interval, we see that there exists a J_ϵ such that $J > J_\epsilon$ implies that $\sum_{j=J}^{\infty} p_{ij}(t) < \epsilon/q_i$. By (1), $\Delta_{ij}(t, t + h) \geqq -q_i p_{ij}(t)$, and hence

$$\sum_{j=J}^{\infty}{}' \Delta_{ij}(t, t + h) > -\epsilon, \qquad h > 0, \qquad J > J_\epsilon, \quad (2)$$

where the prime on the summation indicates a sum over all nonpositive terms (note that the terms involved in the summation vary with t and h).

For given $\epsilon > 0$, let t_1 be any point less than or equal to $\bar{t}$, and let h_n $(n = 1, 2, \ldots)$ be a sequence such that (i) $h_n > 0$, (ii) $h_n \to 0$, and (iii) $|D^+p_{ij}(t_1) - \Delta_{ij}(t_1, t_1 + h_n)| < \epsilon$. By III,

$$\Delta_{ij}(t_1 + s, t_1 + s + h_n) = \sum_{k=J_\epsilon}^{\infty} \Delta_{ik}(t_1, t_1 + h_n)p_{kj}(s) + \Delta_{ij}(t_1 + h_n)p_{jj}(s) + \sum_{\substack{k=1 \\ k\neq j}}^{J_\epsilon - 1} \Delta_{ik}(t_1, t_1 + h_n)p_{kj}(s), \quad (3)$$

where J_ϵ is as in (2) and we assume $J_\epsilon > j$.

Using IV, we can find an η_ϵ such that $p_{kj}(s) < \epsilon/q_i J_\epsilon$ for $k \neq j$, $p_{ij}(s) > (q_i - \epsilon)/q_i$ whenever $0 < s < \eta_\epsilon$, and, applying this to (3), we find that

$$\Delta_{ij}(t_1 + s, t_1 + s + h_n) \geqq -\epsilon + D^+p_{ij}(t_1) - 2\epsilon - \epsilon = D^+p_{ij}(t_1) - 4\epsilon. \quad (4)$$

This implies that $D^+p_{ij}(t_1 + s) \geqq D^+p_{ij}(t_1) - 4\epsilon$ for $0 < s < \eta_\epsilon$ and, therefore, using Dini's theorem, we have that $Dp_{ij}(t_1 + s) \geqq D^+p_{ij}(t_1) - 4\epsilon$ whenever $0 < s < \eta_\epsilon$. From this it follows that $p_{ij}(t_1)$ has a right-hand derivative for $t_1 \in [0, \bar{t}]$ and that

$$D_R p_{ij}(t_1 + s) > D_R p_{ij}(t_1) - \epsilon \qquad \text{for} \qquad 0 \leqq s < \eta_\epsilon. \quad (5)$$

Let t be any point in $[0, \bar{t}]$; it is not difficult to show that $\sum_j D_R p_{ij}(t)$ converges, and, although it is not essential to our argument, we use this result. Let $\sum_j D_R p_{ij}(t) = \alpha$, and take $\epsilon < \alpha/2$ if $\alpha > 0$, and otherwise arbitrary; with J_ϵ as in (2), we can find a $J > J_\epsilon$ and $\delta_1 > 0$ such that

$$\Big|\sum_{j=1}^{J} D_R p_{ij}(t) - \alpha\Big| < \epsilon, \quad (6)$$

$$\Big|\sum_{j=1}^{J} \Delta_{ij}(t, t + h) - \alpha\Big| < \epsilon, \qquad 0 < h < \delta_1. \quad (7)$$

By II, $\sum_{j=1}^{\infty} \Delta_{ij}(t, t + h) = 0$ for any $h > 0$, and, by (2) and (7), $\sum_{j=1}^{\infty} \Delta_{ij}(t, t + h) > \alpha - 2\epsilon$ for $0 < h < \delta_1$; hence, in view of our choice of ϵ, we conclude that $\alpha \leqq 0$. Using II and (7), we see that

$$\Big|\sum_{j=J+1}^{\infty} \Delta_{ij}(t, t + h) + \alpha\Big| < \epsilon, \qquad 0 < h < \delta_1 \quad (8)$$

and, therefore, by (2),

$$\Big|\sum_{j=J+1}^{\infty} |\Delta_{ij}(t, t + h)| + \alpha\Big| < 2\epsilon \qquad \text{for} \qquad 0 < h < \delta_1. \quad (9)$$

Now, by (5), we may choose a $\delta_2 < \delta_1$ such that

$$\sum_{j=1}^{J} D_R p_{ij}(t + h) > \sum_{j=1}^{J} D_R p_{ij}(t) - \epsilon > \alpha - 2\epsilon \qquad \text{for} \qquad 0 \leqq h \leqq \delta_2. \quad (10)$$

From (8) and Dini's theorem it follows that there exists $\bar{J} > J$ and an $\bar{h} < \delta_2$ such that $\sum_{j=J+1}^{\bar{J}} D_R p_{ij}(t + \bar{h}) > -\alpha - \epsilon$, and in view of (2) we obtain

$$\sum_{j=J+1}^{\infty} D_R p_{ij}(t + \bar{h}) > -\alpha - 2\epsilon. \quad (11)$$

Combining (10) and (11), we see that

$$\sum_{j=1}^{\infty} D_R p_{ij}(t + \bar{h}) > -4\epsilon. \tag{12}$$

To summarize, we have shown, in inequalities (9) and (12), that if t' is any point and ϵ any positive number, then there is a point $t'' < t'$ and positive numbers δ and J such that

$$\sum_{j=J+1}^{\infty} |\Delta_{ij}(t'', t'' + h)| < \epsilon \qquad \text{whenever} \qquad 0 < h < \delta. \tag{13}$$

From this, the existence of continuous derivatives follows rather readily; for, by III and (13),

$$|\Delta_{ij}(t' + s, t' + s + h) - \Delta_{ij}(t', t' + h)| = \sum_k \Delta_{ik}(t'', t'' + h)p_{kj}(t' - t'' + s) -$$

$$\sum_k \Delta_{ik}(t'', t'' + h)p_{kj}(t' - t'')| \leqq \Big|\sum_{k=1}^{J} \{\Delta_{ik}(t'', t'' + h)p_{kj}(t' - t'' + s) -$$

$$\Delta_{ik}(t'', t'' + h)p_{kj}(t' - t'')\}\Big| + 2\epsilon \leqq 3\epsilon$$

for s sufficiently small and $0 < h < \delta$. Then, taking the limit as h tends to 0, we see that $|D_R p_{ij}(t' + s) - D_R p_{ij}(t')| \leqq 3\epsilon$; that is, the right derivative of $p_{ij}(t)$ is continuous at t', and hence, by Dini's theorem, $p_{ij}(t)$ has a derivative at t'.

With a little amplification of these arguments the following results may be obtained:

If $q_i < \infty$, then

$$\text{(i)} \quad \sum_j Dp_{ij}(t) = 0, \qquad t > 0,$$

$$\text{(ii)} \quad Dp_{ij}(t_1 + t_2) = \sum_k Dp_{ik}(t_1)p_{kj}(t_2), \qquad t_1 > 0, \qquad t_2 \geqq 0,$$

$$\text{(iii)} \quad Dp_{ij}(t_1 + t_2) = \sum_k p_{ik}(t_1)Dp_{kj}(t_2), \qquad q_j < \infty, \qquad t_2 > 0,$$

$$\text{(iv)} \quad \lim_{h \to 0} \sum_k |\Delta_{ik}(t, t + h)| \leqq 2q_i.$$

[1] J. L. Doob, "Topics in the Theory of Markoff Chains," *Trans. Am. Math. Soc.*, **52**, 37–64, 1942.

[2] A. N. Kolmogoroff, *Uchenye Zapiski (Mat.) Moskov. Gosudarst. Univ.*, **148**, 53–59, 1951.

ON SOME NONLINEAR INTEGRAL EQUATIONS OCCURRING IN THE THEORY OF DYNAMIC PROGRAMMING

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1. *Introduction.*—The purpose of this paper is to present a summary of some results we have recently obtained concerning the nonlinear integral equation

$$f(x) = \operatorname*{Min}_{y \geq x} [k(y - x) + a[\int_y^\infty p(s - y)\, dG(s) + f(0) \int_y^\infty dG(s) + \int_0^y f(y - s)\, dG(s)]] \quad (1.1)$$

and its extensions in various directions. Equations of this type occur in connection with the problem of optimal inventory in mathematical economics and are typical of some classes of functional equations which arise in the theory of dynamic programming.[1] The mathematical formulation of equation (1.1) was first given by Arrow;[2] a more detailed study containing existence and uniqueness theorems and a discussion of some related statistical questions is given by Dvoretzky *et al.*;[3] cf. also Bellman.[4]

Up to the present time, however, no general analytic solution of equation (1.1) has been given, and it would seem that a solution for arbitrary functions must possess a quite complicated structure. Here we shall present a number of solutions for the case where $k(z)$ and $p(z)$ are linear functions of z, for various classes of distribution functions. A detailed account will appear elsewhere.

2. *The Basic Analytic Device.*—Let us now indicate briefly the essence of the method we have employed. The key to the mathematical analysis is the observation that, in the equation

$$u(x) = \operatorname*{Min}_{y \geq x} v(x, y) \quad (2.1)$$

in an x-interval where the minimum occurs at finite values $y > x$, we have in place of equation (2.1) the simultaneous equations

$$v_y(x, y) = 0, \qquad u'(x) = v_x(x, y). \quad (2.2)$$

This observation, combined with a strenous application of the method of successive approximations, yields all the results enunciated below.

3. *"Constant Stock Level."*—We shall now present a number of results linked by a common feature of the solution. Our first result is

Theorem 1. *Consider equation* (1.1) *above under the following assumptions:*

$$\begin{aligned} &a) \quad k(z) = kz, \qquad p(z) = pz, \qquad k, p > 0; \\ &b) \quad dG(s) = \phi(s)ds, \qquad \phi > 0; \\ &c) \quad 0 < a < 1, \qquad ap > k. \end{aligned} \quad (3.1)$$

Let $\bar{x}$ be the unique root of the equation

$$k = ap \int_y^\infty \phi(s)\, ds + ak \int_0^y \phi(s)\, ds. \quad (3.2)$$

Then the solution to equation (1.1) *is furnished by the prescription*

$$\begin{aligned} &a) \quad y = \bar{x} \text{ for } 0 \leq x \leq \bar{x}, \\ &b) \quad y = x \text{ for } x > \bar{x}. \end{aligned} \tag{3.3}$$

If $ap \leq k$, *the solution is* $y = x$ *for* $x \geq 0$.

This result generalizes to the multidimensional versions of equation (1.1). For two dimensions the result reads:

THEOREM 2. *Consider the equation*

$$f(x_1, x_2) = \underset{\substack{y_1 \geq x_1 \\ y_2 \geq x_2}}{Min} [k_1(y_1 - x_1) + k_2(y_2 - x_2) + a[\int_{y_1}^{\infty} \int_{y_2}^{\infty} [p_1(s_1 - y_1) + p_2(s_2 - y_2) + f(0, 0)] \, dG(s_1, s_2) + \int_{y_1}^{\infty} \int_{0}^{y_2} [p_1(s_1 - y_1) + f(0, y_2 - s_2)] \, dG(s_1, s_2) + \int_{0}^{y_1} \int_{y_2}^{\infty} [f(y_1 - s_1, 0) + p_2(s_2 - y_2)] \, dG(s_1, s_2) + \int_{0}^{y_1} \int_{0}^{y_2} f(y_1 - s_1, y_2 - s_2) \, dG(s_1, s_2)]],$$

where we assume that

$$\begin{aligned} &a) \quad k_i(y) = k_i y, \qquad p_i(y) = p_i y, \qquad \text{where } k_i \text{ and } p_i \text{ are positive constants;} \\ &b) \quad dG(s_1, s_2) = \phi(s_1, s_2) \, ds_1 \, ds_2, \qquad \phi > 0; \\ &c) \quad \int_0^{\infty} \int_0^{\infty} s_i \phi(s_1, s_2) \, ds_1 ds_2 < \infty, \qquad i = 1, 2; \\ &d) \quad 0 < a < 1, \qquad ap_i > k_i. \end{aligned} \tag{3.5}$$

Then the solution to equation (3.4) *is as follows: Let* $\bar{x}_i$, $i = 1, 2$, *be defined as the unique positive solutions of the equations*

$$\begin{aligned} k_1 - p_1 a \int_{\bar{x}_1}^{\infty} (\int_0^{\infty} \phi(s_1, s_2) \, ds_2) \, ds_1 - k_1 a \int_0^{\bar{x}_1} (\int_0^{\infty} \phi(s_1, s_2) \, ds_2) \, ds_1 = 0, \\ k_2 - p_2 a \int_{\bar{x}_2}^{\infty} (\int_0^{\infty} \phi(s_1, s_2) \, ds_1) \, ds_2 - k_2 a \int_0^{\bar{x}_2} (\int_0^{\infty} \phi(s_1, s_2) \, ds_1) \, ds_2 = 0. \end{aligned} \tag{3.6}$$

Then

$$\begin{aligned} &a) \quad \text{For } 0 \leq x_i \leq \bar{x}_i \quad \text{we choose } y_i = \bar{x}_i, \\ &b) \quad \text{For } x_i > \bar{x}_i \quad \text{we choose } y_i = x_i. \end{aligned}$$

Observe that the form of the solution is independent of the dimension and has the important property of suboptimality. There are analogous results for the general case where G does not possess a density. We have omitted these because of the additional complication due to lack of uniqueness of the solutions of equations (3.6).

THEOREM 3. *Consider equation* (1.1), *where we take* $p(z) = pz + q$, $q > 0$, *and assume that the other conditions hold. If* ϕ *is continuous and the function*

$$F'(y) = k + a[-p \int_y^{\infty} \phi(s) \, ds - q\phi(y) - k \int_0^y \phi(s) \, ds] = 0$$

has a unique root $y = \bar{x}$ *for* $0 \leq y < \infty$, *the solution will be as before. If it has a multiplicity of roots and if the largest positive root* $y = \bar{x}$ *corresponds to the absolute minimum of the function*

$$F(y) = ky + a[\int_y^{\infty} [p(s - y) + q]\phi(s) \, ds - k \int_0^y (y - s)\phi(s) \, ds],$$

then the solution is still as before.

Some particular cases where the above holds are furnished by the familiar density functions

$$a) \quad \phi(x) = \frac{e^{-(x-a)^2}}{\int_a^\infty e^{-u^2}\,du}, \qquad x \geq 0;$$
$$b) \quad \phi(x) = be^{-bx}, \qquad x \geq 0; \tag{3.7}$$

4. *More Complicated Solutions.*—Let us now present some solutions of more complicated type.

THEOREM 4. *Consider the equation*

$$f(x) = \min_{z \geq 0} \left[kz + a\left[\int_x^\infty p(s-x)\phi(s)\,ds + f(z)\int_x^\infty \phi(s)\,ds + \int_0^x f(x-s+z)\phi(s)\,ds\right]\right], \tag{4.1}$$

under the preceding assumptions.

The optimal policy is determined by a function $z(x)$, which is monotone-decreasing, and a constant $\bar{x}$,

$$\begin{aligned} &a) \quad z = z(x) \text{ for } 0 \leq x \leq \bar{x}, \\ &b) \quad z = 0 \quad \text{for } x \geq \bar{x}, \end{aligned} \tag{4.2}$$

where $z(\bar{x}) = 0$.

THEOREM 5. *Consider equation (1.1), where $k(z)$ is now a convex monotone-increasing function of z, and the other conditions hold. The optimal policy is determined by a function $y(x)$ and a constant $\bar{x}$. If $ap > k'(0)$, $\bar{x} > 0$; otherwise, $\bar{x} = 0$. The function $y(x)$ is monotone-decreasing, and $y(x) \geq x$.*

$$\begin{aligned} &a) \quad y(x) > x \text{ for } 0 \leq x \leq \bar{x}, \\ &b) \quad y(x) = x \text{ for } x \geq \bar{x}. \end{aligned}$$

[1] R. Bellman, "An Introduction to the Theory of Dynamic Programming" (RAND Report No. 245, 1953).

[2] K. J. Arrow, T. E. Harris, and J. Marschak, "Optimal Inventory Policy," *Econometrica*, July, 1952.

[3] A. J. Dvoretzky, J. Kiefer, and J. Wolfowitz, "The Inventory Problem. I and II," *Econometrica*, 1952, pp. 187–222.

[4] R. Bellman, "The Theory of Dynamic Programming," *Bull. Am. Math. Soc.*, **60**, 503–516, 1954.

CONVERGENCE ALMOST EVERYWHERE OF OPERATOR AVERAGES

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Let μ be a nonnegative countably additive set function defined on the σ-field Σ of subsets of a set S, and, for $1 \leq p \leq \infty$, let $|f|_p$ denote the norm of an element in the Lebesgue space $L_p = L_p(S, \Sigma, \mu)$; for a measurable function f on S which is not in L_p let $|f|_p = \infty$. If T is a linear operator with domain $D(T) \supseteq L_1 \cap L_\infty$ and range $R(T) \subseteq L_1 \cup L_\infty$, let $|T|_p$ denote the smallest K such that $|Tf|_p \leq K|f|_p$ for $f \in D(T)$. Then the following two theorems constitute the substance of the present report.

THEOREM 1. *Let $\{T_i(t)\}$, $i = 1, \ldots, k$, be a set of strongly measurable semigroups of operators in L_1, such that $|T_i(t)|_1 \leq 1$, $|T_i(t)|_\infty \leq 1$, $i = 1, \ldots, k$, $t > 0$. For each $\alpha_1, \ldots, \alpha_k > 0$ and each $f \in L_p$ let*

$$A(\alpha_1, \ldots, \alpha_k)f = (\alpha_1 \ldots \alpha_k)^{-1} \int_0^{\alpha_1} \ldots \int_0^{\alpha_k} T_1(t_1) \ldots T_k(t_k) f dt_1 \ldots dt_k.$$

Then, for $p > 1$ and $f \in L_p$, the limit (as $\alpha_1, \ldots, \alpha_k$ approach ∞ independently) of $A(\alpha_1, \ldots, \alpha_k)f$ exists almost everywhere. Moreover, there exists a function $f^* \in L_p$ dominating all the functions $A(\alpha_1, \ldots, \alpha_k)f$, $\alpha_i > 0$.

THEOREM 2. *Let $\{T(t_1, \ldots, t_k)\}$ be a k-parameter strongly measurable semigroup of operators in L_1, such that $|T(t_1, \ldots, t_k)|_1 \leq 1$ and $|T(t_1, \ldots, t_k)|_\infty \leq 1$ for $t_i > 0$, $i = 1, \ldots, k$. For each $\alpha > 0$ and each $f \in L_1$ put*

$$A(\alpha)f = \alpha^{-k} \int_0^\alpha \ldots \int_0^\alpha T(t_1, \ldots, t_k) f dt_1 \ldots dt_k.$$

Then, for each $f \in L_1$, $\lim_{\alpha \to \infty} A(\alpha)f$ exists almost everywhere.

The discrete analogues of both these theorems are also true. Theorem 1 can be deduced from its particular case $k = 1$ by an argument similar to that given by Dunford.[1] A technical argument will reduce this case to its discrete analogue. The following lemma, which is of some independent interest, allows us to introduce, without loss of generality, the additional hypothesis that the linear operators under consideration preserve positivity of functions.

LEMMA 1. *Let T be an operator in L_1 such that $|T|_1 < \infty$, $|T|_\infty < \infty$. Then there exists a positivity-preserving operator P such that $|P|_1 = |T|_1$, $|P|_\infty = |T|_\infty$, and $|Tf(\cdot)| \leq P|f(\cdot)|$ for each $f \in L_1$.*

The proof may be made by defining, for positive f, $Pf = \sup_{|g(\ )| < f} |Tg(\cdot)|$. Then each of the properties stated for P can be verified either by an elementary argument or by an argument making use of L_1, its conjugate space, and its second conjugate space.

The reductions outlined above reduce Theorem 1 to its discrete one-parameter case, which, in turn, may be derived without difficulty from the following lemma, where is essentially due to E. Hopf.[2]

LEMMA 2. *Let T be a positivity-preserving operator in L_1, with $|T|_1 \leq 1$. Let f be a real function in L_1, with $\sup_n n^{-1} \sum_{j=0}^{n-1} T^j f \geq 0$. Then $\int_S f(s)\mu(ds) \geq 0$.*

Theorem 2 is reduced, via a detour through the discrete case and by use of Lemma 1, to the case in which all the operators $\{T(t_1, \ldots, t_k)\}$ preserve positivity. This case is then treated by induction on k. Thus, for instance, consideration of the two-parameter semigroup $\{T(t_1, t_2)\}$ is reduced to consideration of the one-parameter semigroup

$$S(x) = (4\pi)^{-1}x^2 \int_0^\infty \int_0^\infty (t_1t_2)^{-3/2} \exp(t_1x^{-2}) \exp(t_2x^{-2}) T(t_1, t_2)\, dt_1\, dt_2.$$

The discrete analogue of Theorem 2 is reduced to the continuous case. Consideration of the two-parameter discrete semigroup $\{T_1^{i_1}T_2^{i_2}\}$ is, for instance, reduced to consideration of the two-parameter continuous semigroup

$$\exp(t_1(T_1 - I) + t_2(T_2 - I)).$$

* This paper was prepared under contract NONR609(04) with the Office of Naval Research.

[1] N. Dunford, "An Individual Ergodic Theorem for Noncommutative Transformations," *Acta Szeged*, **14**, 1–4 1951.

[2] E. Hopf, "The General Temporally Discrete Markoff Process," *J. Rat. Mech. Anal.*, **3**, 13–45, 1954.

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*UNIFORMLY BOUNDED REPRESENTATIONS OF GROUPS**

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It has been conjectured for a long time that if a representation $B: g \to B(g)$ of a group G by bounded operators $B(g)$ on a Hilbert space H is uniformly bounded, then B should be equivalent to a unitary representation of G. For many classes of groups this has been proved to be correct. It will now be shown that there exist very important groups for which the conjecture is false.

Let G be the group of real 2×2 matrices of determinant 1. For any complex-valued function $f(x)$ of the real variable x, any complex number $s = \sigma + it$, and any element g of G we put

$$[M(g, s)f](x) = \left[\frac{dgx}{dx}\right]^s f(gx) = |bx + d|^{-2s} f\left(\frac{ax + c}{bx + d}\right), \tag{1}$$

where g is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let H_σ be the completion in the norm (2) of the space of all equivalence classes of those measurable functions $f(x)$ for which

$$||f||_\sigma^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x - y|^{-2\sigma} f(x)\overline{f(y)}\, dx\, dy \tag{2}$$

is finite. Since $|x|^{-2\sigma}$ is a positive definite function on the additive group of real numbers provided that

$$0 < \sigma < 1/2, \tag{3}$$

H_σ is for such values of σ a Hilbert space with the inner product

$$\langle f_1, f_2 \rangle_\sigma = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x - y|^{-2\sigma} f_1(x)\overline{f_2(y)}\, dx\, dy. \tag{4}$$

It is known that if $0 < \sigma < 1/2$, then $g \to M(g, \sigma)$ is an irreducible unitary representation of G with representation space H_σ, the so-called exceptional interval of irreducible unitary representations of G discovered by Bargmann.[1]

We shall now show that if s is any fixed complex number whose real part $Rs = \sigma$ satisfies condition (3), then the bound $|||M(g, s)|||$ of the operator $M(g, s)$ is uniformly bounded in g. Let, at first, g be the diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ with $a > 0$; then

$$[M(g, s)f](x) = a^{2s} f(a^2 x);$$

hence

$$||M(g, s)f||_\sigma^2 = \int\int |x - y|^{-2\sigma} a^{2s} f(a^2x)\overline{a^{2s} f(a^2y)}\, dx\, dy$$

$$= \int\int |x - y|^{-2\sigma} a^{2\sigma} f(a^2x)\overline{a^{2\sigma} f(a^2y)}\, dx\, dy = ||M(g, \sigma)f||_\sigma^2.$$

Since $M(g, \sigma)$ is a unitary operator of the space H_σ for $0 < \sigma < 1/2$, we have $||M(g, \sigma)f||_\sigma \leqq 1$ whenever $||f||_\sigma \leqq 1$; this proves that

$$||M(g, s)f||_\sigma \leqq 1 \text{ whenever } ||f||_\sigma \leqq 1,\ 0 < \sigma < 1/2, \text{ and } g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix},\ a > 0.$$

It will be shown elsewhere that $M(k, s)$ is a bounded operator on $H\sigma$ for each $k \in K$. Hence if g varies over the compact subgroup K of rotations, then the bound $|||M(g, s)|||$ is certainly a bounded function of g on K. Since every element of G is the product of two orthogonal matrices and one diagonal matrix with positive characteristic roots, it follows that $|||M(g, s)|||_\sigma$ is a bounded function on G for any fixed complex s with $0 < \sigma < 1/2$.

It remains to show that if s is not real, then the representation $g \to M(g, s)$ is not equivalent to a unitary representation of G. Let $D(G)$ denote the algebra of all indefinitely differentiable functions on G which have compact supports. Let J_s be the set of those elements $x(g) \in D(G)$ for which $\int x(g)M(g, s)\, dg = 0$, where dg refers to Haar measure on G. Let J_s^0 be the set of spherical functions contained in J_s, i.e., J_s^0 consists of those elements $x(g) \in J_s$ which satisfy $x(kgk') = x(g)$ for all $k, k' \in K$, $g \in G$. Using Bargmann's description[2] of the Hilbert space H_σ, it follows that $x(g) \in J_s^0$ if and only if

$$\int x(g)\varphi(g, s)\, dg = 0, \tag{5}$$

where

$$\varphi(g, s) = \int_0^1 \left[\frac{d\tilde{g}\,\theta}{d\theta}\right]^s d\theta \tag{6}$$

is the "elementary spherical function" for the value s, and $\tilde{g}$ the Caley transform of g (so that $\tilde{g}$ acts on the unit circle). Now consider the adjoint $x^*(g) = \bar{x}(g^{-1})$ of $x(g)$. It is easy to see that J_s^0 is self-adjoint, i.e., $(J_s^0)^* = J_s^0$ if and only if $\varphi(g, s) = \varphi^*(g, s)$ for this s. On the other hand, it is known[3] that $\varphi(g, s) = \varphi^*(g, s)$ if and only if $Rs = 1/2$ or s is real. Hence, if s satisfies $0 < \sigma < 1/2$, Im $s \neq 0$, then the set J_s^0 is not self-adjoint. From this one concludes that J_s is not self-adjoint. Hence there cannot exist a bounded operator C_s of the Hilbert space H_σ which has a bounded inverse C_s^{-1} such that $g \to C_s^{-1}M(g, s)C_s$ is a unitary representation of G. This completes the proof.

It should be noted that this argument shows that for $0 < \sigma < 1/2$ the representation $g \to M(g, s)$ of G is not equivalent to a unitary representation even in the a priori wider sense of calling two representations equivalent if their kernels in $D(G)$ are the same.

It follows from results of the authors[3] that the sets J_s^0 and $J_{s'}^0$ are different, provided that $s' \neq s$, or $1 - s$. This shows that the representations $M(g, s)$ and $M(g, s')$ are not equivalent provided that $s' \neq s$, or $1 - s$. We conclude by observing that the above group G of real 2×2 matrices is by no means an isolated

counterexample; essentially the same argument applies to the complex unimodular group, for example. The more general case of arbitrary noncompact semi-simple Lie groups will be discussed in a later publication.

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[1] V. Bargmann, "Unitary Representations of the Lorentz group," *Ann. Math.*, **48**, 568–640, 1947.

[2] *Ibid.*, pp. 618–620.

[3] L. Ehrenpreis and F. I. Mautner, "Some Properties of the Fourier Transform on Semi-simple Lie Groups" *Ann. Math.* (to appear).

ON CANONICAL CONSTRUCTIONS. I

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Given a structure S and a structure S' derived from S, the question frequently arises, but is infrequently recognized, of when S may be reconstructed from S'. A large class of theorems depends on the fact that in certain special instances such a reconstruction may be effected canonically. In order that they may serve as paradigms, the proofs of certain classical theorems will be examined in these papers to show how a canonical construction properly plays a basic role. The first theorems to be discussed will be those having to do with inner automorphisms in the sense in which they are here defined.

For the present, the concepts of structure, substructure, structure derived from another structure, isomorphism of one structure onto or into another, and an automorphism of a structure will be assumed to be at least intuitively clear. In speaking of a structure derived from another, it is assumed that to every structure of a kind defined by some system of axioms there is assigned in an intrinsic manner a structure defined by the same or some other system. The statements on structures which follow may be taken as axiomatic or intuitive; their validity in any instance is easily verified.

A canonical correspondence is one which can be exhibited without "arbitrary choice." It is the unique correspondence satisfying certain conditions, and the definition of "canonical" is relative to these conditions. Given a structure T, substructures S_1, S_2, an automorphism σ of T such that $\sigma(S_1) = \sigma(S_2)$, and isomorphisms ϕ_1 and ϕ_2 of some other structure S onto S_1 and S_2, respectively, then, if ϕ_1 and ϕ_2 are distinct, neither is canonical.

If S, T are structures, S', T' structures derived from them, and ϕ an isomorphism of S onto T, then there exists a unique (hence canonical) isomorphism ϕ' of the structure (S, S') consisting of S, S', and the relations between them (i.e., the manner of derivation) onto the structure (T, T'), which extends ϕ. The isomorphism ϕ' will be called "the isomorphism induced by ϕ." It may be restricted to an isomorphism of T onto T' and will frequently be so considered. From the preceding one obtains

PROPOSITION 1. *Let S be a structure, S' a structure derived from S, and let there be a canonical isomorphism of S onto a substructure Q of S'. Let T be a structure isomorphic to S and T' be the structure derived from T in the same manner as S' from S. Then there exists a canonical isomorphism of T onto a substructure R of T' such that, if ϕ is any isomorphism of S onto T and ϕ' the isomorphism it induces of (S, S') onto (T, T'), then $\phi'(Q) = R$, and the canonical isomorphism of T onto R is the image under ϕ' of that of S onto Q.*

The composition of canonical isomorphisms is canonical, and the isomorphism induced by a product of isomorphisms is the product of the induced isomorphisms. Let S be a structure and S' one derived from S. An automorphism of S induces one of S'. Such an induced automorphism will be called "inner." The automorphisms of a structure S form a group. There is a canonical homomorphism from this group onto the group of inner automorphisms of any structure S' derived from S. The concept of inner automorphism is relative, referring to an underlying structure. If S' is itself a group, then an inner automorphism of S' as a derived structure need not be an inner automorphism of S' as a group; likewise the converse is not generally true. However, if S' is the group of all automorphisms of a structure S, then the concepts of inner automorphism of S' as a derived structure and as a group coincide, and every group can, in fact, be represented as the full group of automorphisms of a suitable structure. From Proposition 1 follows

PROPOSITION 2. *If S is a structure, S' a structure derived from S, and Q a substructure of S' which is canonically isomorphic to S, then every automorphism of Q may be extended uniquely to an inner automorphism of S' (relative to the derivation of S' from S).*

Given structures S and T, if S is canonically isomorphic to a structure T' derived from T, then we shall say that S is canonically reconstructible from T. The preceding proposition may then be put in the following form:

SCHOLIUM[1] 1. *If a structure S contains a substructure T, and if S is canonically reconstructible from T, then every automorphism of T may be extended uniquely to an inner automorphism of S (relative to the derivation of S from T).*

Given a structure S' derived from a structure S, we may consider S' as a substructure of (S, S'). The resulting special case of Scholium 1 is the following:

SCHOLIUM[1] 2. *Given a structure S and a structure S' derived from S, if S is canonically reconstructible from S', then every automorphism of S' is inner.*

For want of space, only the simplest example illustrating Scholium 2 will be given here. In the simplest case, S is a set and S' the group of all automorphisms, i.e., permutations, of S. This group will be called the "symmetric group on S," or "on n elements" if n is the cardinality of S. A theorem of Hölder[2] asserts that every automorphism of the symmetric group on n elements is inner if n is finite and different from six. It will be shown that, for every finite or infinite n different from six, S may be reconstructed canonically from the symmetric group on S. The truth of Hölder's theorem, and its generalization to the infinite case, will then follow immediately from Scholium 2. By the same techniques it can, in fact, be shown that if S is a set of any finite or infinite cardinality different from three and six, S' the symmetric group on S, and G any subgroup of S' containing all three-cycles, then every automorphism of G can be extended uniquely to an inner

automorphism of S'. This extends another theorem of Hölder[3] and will be done in a later paper.

The case in which S is a topological space will be discussed in the second paper in this series, which will contain a proof of the theorem of Fine and Schweigert[4] asserting that every automorphism of the group of homeomorphisms of the unit interval onto itself is inner.

The Theorem of Hölder.—Let S be a set of at least three elements (otherwise, all considerations are trivial) and S' be the symmetric group on S. A transposition will be written in the usual way: (i, j). Two transpositions fail to commute if and only if they have exactly one index in common. Three transpositions have exactly one index in common if and only if no two commute and the third is not the conjugate of the first by the second. It follows that the elements of S are in canonical one-one correspondence with the maximal sets of transpositions such that no two elements of the set commute and of no three elements is it the case that the third is the conjugate of the first by the second. Therefore, S will be reconstructed and the generalized theorem of Hölder proved if a transposition can be axiomatically characterized within the structure of S'. This is possible except when the cardinality of S is six.

A transposition has the following properties: (1) It is of order two. (2) The set of all elements of S' commuting with it is a maximal subgroup. (3) If S' is finite and has $n!$ elements, then the number of elements commuting with any transposition is $(n - 2)!2$. It will be shown that, except when $n = 6$, these properties characterize a transposition.

Any permutation of order two is a finite or formal infinite product of disjoint (commuting) transpositions. An element of order two which moves every element of S will be called a "revolution." An element of order two such that the set of all elements of S' commuting with it is a maximal subgroup is either a transposition or revolution, for the group of all permutations commuting with a permutation containing the product of two disjoint transpositions and leaving fixed some element of S is certainly not maximal. When S is infinite, the subgroup of S' consisting of all permutations commuting with a revolution is, in fact, not maximal, since to it may be adjoined any transposition without obtaining the full symmetric group. Therefore, if S' is infinite, a transposition may be characterized by the fact that it is of order two and the set of all elements of S' commuting with it is a maximal subgroup, ending the proof of the theorem of Hölder in the infinite case. If n is finite, the previous argument fails, but to complete the characterization of a transposition it is sufficient to show that the number of elements commuting with a revolution (which can exist only if n is even) differs from the number commuting with a transposition except when $n = 6$. The former number is $(n/2)!2^{n/2}$, which certainly differs from $(n - 2)!2$ when $n \neq 6$. This ends the proof of the theorem of Hölder.

The author is much indebted to Professors Fine, Herstein, Schweigert, and Whaples and particularly to Dr. Zager for helpful suggestions.

[1] A scholium is an epexegetical remark. The author suggests that the word be used to denote a metamathematical argument, applied (as these will be) in a mathematical discussion.

[2] O. Hölder, "Bildung zusammnengesetzer Gruppen," *Math. Ann.*, **46**, 1895, esp. 340–345. A short proof is given by I. E. Segal, "The Automorphisms of the Symmetric Group," *Bull. Am. Math. Soc.*, **46**, 565, 1940.

[3] Hölder, *op. cit.*

[4] N. Fine and G. Schweigert, "On the Group of Homeomorphisms of an Arc," *Ann. Math.* (to appear). This theorem should be a prototype for many others. Analogous theorems for other topological spaces will be called "Fine-Schweigert theorems." The author has established the Fine-Schweigert theorem for the disk and will give its proof in another paper in this series.

ON THE MODULI OF RIEMANN SURFACES

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In notes[1] in these PROCEEDINGS I have shown that, for an algebraic Riemann surface S which is not hyperelliptic, those periods π_{ij} of the normal integrals of the first kind, $\int d\zeta_i$, whose indices, i, j, are those for which the products $d\zeta_i\, d\zeta_j$ form a basis of the quadratic differentials $d\zeta^2$ on S constitute a set of numerical moduli and also that an analogous theorem holds for plane domains whose doubles are not hyperelliptic.

Now, on a hyperelliptic surface S of genus p the $d\zeta^2$ are not all linear combinations of products of simple differentials. In fact, as I will show in a moment, only $2p - 1$ $d\zeta^2$ are of this form, while the remaining $p - 2$ are not. Thus, as pointed out in the first note, a second S' might very well have its periods π'_{ij} equal to π_{ij} for those indices for which $d\zeta_i\, d\zeta_j$ generate the $2p - 1$ $d\zeta^2$ of this form, without being conformally equivalent to S. *If, however, S' is assumed to be hyperelliptic, too*, then I claim that they are indeed conformally equivalent. That is foreshadowed by the fact, already known,[2] that the hyperelliptic surfaces as a class depend on $2p - 1$ parameters.

THEOREM 1. *Let S be a hyperelliptic Riemann surface of genus p with normal integrals of the first kind, $\int d\zeta_i$, with periods π_{ij} over the retrosections, δ_j, $i, j = 1, \ldots, p$. Let $d\zeta_i\, d\zeta_j$, $(i, j) \in (I, J)$, form a basis for those $d\zeta^2$ on S which are linear combinations of products of simple differentials. Then, if a second hyperelliptic surface, S', of genus p with periods π_{ij}, etc., is such that*

$$\pi_{ij} = \pi_{ij}', \qquad (i, j) \in (I, J), \tag{1}$$

then S' can be mapped conformally and 1-1 *on S.*

To establish Theorem 1, I need merely show the slight modifications of the proof of Theorem 1 of the first note required here.

First of all, a hyperelliptic surface is, by definition, one which admits a representation as the two-sheeted surface of $y^2 = f(x)$, where $f(x)$ is a polynomial of degree $2p + 2$, if $x = \infty$ is not a branch-point, and $2p + 1$ if it is. Then the $2p - 2$ simple differentials of the first kind, i.e., everywhere finite, are

$$\frac{dx}{y}, \frac{x\,dx}{y}, \ldots, \frac{x^{p-1}\,dx}{y}. \tag{2}$$

All possible products of two of these are the $2p - 1$ finite quadratic differentials

$$\frac{dx^2}{y^2}, \frac{x\,dx^2}{y^2}, \ldots, \frac{x^{2p-1}\,dx^2}{y^2}. \tag{3}$$

But, besides (3), there are still the $p - 2$ $d\zeta^2$ which are *not* products of (3),

$$\frac{dx^2}{y}, \frac{x\,dx^2}{y}, \ldots, \frac{x^{p-3}\,dx^2}{y}. \tag{4}$$

Another way to distinguish (3) from (4) is to observe that S and S' necessarily admit the conformal involutions

$$T: \quad y \to -y \text{ and } T': \quad y' \to -y',$$

i.e., the exchange of the two sheets. Under T, (3) are invariant; (4) are not.

Now, in minimizing the Dirichlet functional $J(\phi)$ among mappings ϕ of S onto S', let the ϕ be confined to those which are not only 1-1 and deformable into a fixed ϕ_0 but which also admit T, i.e., for which $\phi\,(TP) = T'\phi(P)$, $P \epsilon S$. To put it another way, ϕ must carry symmetric points into symmetric points.

To express this property of admissible mappings analytically, I observe that, if z is a local parameter at $P \epsilon S$, then it can also be used at TP simply by projecting one sheet on the other (at branch-points $P = T(P)$). Similarly, w at $P' = \phi(P)$ can be used at $T'P'$. Thus, if one rewrites $P' = \phi(P)$ as

$$w = \phi(z, \bar{z}), \tag{5}$$

then (5) holds both for P and P' and TP and $T'P'$. Hence in the decomposition

$$dw\,d\bar{w} = b\,dz\,d\bar{z} + 2\,\mathrm{Re}\,(a\,dz^2) \tag{6}$$

one sees that $a\,dz^2$ has the same value at TP as at P, i.e., must be a linear combination of (3) *alone* and hence of $d\zeta_i\,d\zeta_j$ but not of (4).

Hypothesis (1), as in the first note, then implies that $a\,dz^2$ is orthogonal to every $d\zeta_i\,d\zeta_j$ and hence to (3) and, therefore, is identically zero, so that the conclusion follows.

THEOREM 2. *Let D be a plane domain bounded by nonintersecting Jordan curves C_i, $i = 1, \ldots, n$, having harmonic measures ω_i, $i = 1, \ldots, n - 1$, and associated analytic functions $w_i = \omega_i + i\mu_i$. Let the double, S, of D be hyperelliptic, and let $(\partial w_i/\partial z)\,(\partial w_j/\partial z)\,dz^2$, $(i, j) \epsilon (I, J)$, form a complex-independent basis on S for the $d\zeta^2$ of the form $d\zeta_i\,d\zeta_j$ and thus a real-independent basis on D. Let the period of the i-th harmonic measure of D over C_j be P_{ij}. If a similar domain D' with ω_i' etc., is such that*

$$P_{ij} = P'_{ij}, \qquad (i, j) \epsilon (I, J), \tag{7}$$

then D' can be mapped conformally and 1-1 *on D.*

This follows from the second note in the same fashion in which Theorem 1 follows from the first note. In fact, if S is hyperelliptic, then not only S but also S admits a conformal involution T, as shown in Lemma 1 of the second note. Hence the mappings of D on D' can again be chosen to admit T. T can be visualized by mapping D on the upper half-plane so that C_n goes into the real axis, while C_ν

$i = 1, \ldots, n-1$, go into circles, $\tilde{C}_i$, where the center of C_1 is fixed and the center of $\tilde{C}_2$ is constrained to lie on a vertical line L through the center of C_1. T is the reflection in L, and the set of D admitting it[3] consists of those for which the centers of all $\tilde{C}_i$ lie on L, thus depending on $n-2$ centers and $n-1$ radii $= 2n-3 = 2p-1$ parameters.

[1] On the Transcendental Moduli of Algebraic Riemann Surfaces" (hereafter referred to as the "first note"), these PROCEEDINGS, **41**, 42, 1955; "On Moduli in Conformal Mapping" (hereafter referred to as the "second note") these PROCEEDINGS, **41**, 176, 1955. The notation in these notes is carried over intact.

[2] Severi-Löffler, *Vorlesungen über algebraische Geometrie* (Leipzig, 1921).

[3] That is, those which admit T and are also of the type having hyperelliptic double—obviously there are others admitting T.

NONANALYTIC FUNCTIONS OF ABSOLUTELY CONVERGENT FOURIER SERIES

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We denote by A the set of all complex-valued functions f of the form

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \qquad (-\pi < \theta \leq \pi)$$

with $\sum |c_n| < \infty$. The subset A^+ of A consists of all f in A for which $c_n = 0$ whenever $n < 0$. With every f in A^+ there is associated an absolutely convergent power series

$$F(z) = \sum_{-\infty}^{\infty} c_n z^n \qquad (|z| \leq 1)$$

such that $F(e^{i\theta}) = f(\theta)$.

A famous theorem of Wiener[1] and Lévy[2] asserts that, if f belongs to A and if ϕ is a function which is analytic at every point of the range[3] of f, then the composite function[4] $\phi \circ f$ also belongs to A. The principal result of this note (Theorem 2) shows that the conclusion of the Wiener-Lévy theorem may fail to hold even if there is only one point in the range of f at which ϕ is merely continuous and not analytic. The special case in which ϕ is the square root leads to the impossibility of certain factorizations in the convolution algebra L_1.

Our method is geometric and requires no computation. It depends on the Riemann mapping theorem and on the following two relations between rectifiability and absolute convergence of power series on the closed unit disk:

THEOREM 1. *Let* $F(z) = \sum_{0}^{\infty} c_n z^n$ *be analytic in* $|z| < 1$ *and continuous on* $|z| \leq 1$.

(i) *If* $\sum |c_n| < \infty$, *then* F *maps every radius of the unit disk into a rectifiable curve.*

(ii) *If F maps the unit circumference into a rectifiable curve, then* $\sum |c_n| < \infty$.

Part (i) is trivial, since

$$\int_0^1 |F'(re^{i\theta})|\, dr \leq \sum_1^\infty n|c_n| \int_0^1 r^{n-1}\, dr = \sum_1^\infty |c_n|.$$

Part (ii) is due to Hardy and Littlewood.[5]

THEOREM 2. *Let ϕ be continuous on the closure of a plane domain D and analytic in D. Suppose that t is a boundary point of D, and L is a rectifiable arc which has t as one end-point but lies otherwise in D, such that the derivative $\phi'(w)$ is unbounded for w on $L \cap D$. Then there exists a function f in A^+ with the following properties: (1) $f(\theta)$ is in D if $\theta \neq 0$; $f(0) = t$. (2) $\phi \circ f$ does not belong to A.*

Proof: Choose a sequence of points w_n ($n = 1, 2, 3, \ldots$) on $L \cap D$ such that $|\phi'(w_n)| > n$. Replace L by a rectifiable polygonal arc M through the points w_n, such that t is the only limit point of the vertices of M. Then, for each n, we can clearly find a closed circular disk K_n in D with center at w_n and diameter less than n^{-2}, such that $|\phi'(w)| > n$ on K_n, such that the intersection of K_n and M is an arc with end-points a_n and b_n, and such that K_n and K_m do not intersect if $n \neq m$.

Let A_n, B_n be disjoint arcs on the boundary of K_n, containing a_n and b_n, respectively, and let T_n be a topological 2-cell[6] with the following properties: T_n contains A_n and B_n but is otherwise in the interior of K_n; the boundary of T_n is rectifiable, its length being less than $3n^{-2}$; the length of any arc in T_n joining a point of A_n and a point of B_n is at least n^{-2}.

We can now form a simply connected domain R which contains $M \cap D$ and has the following properties: the intersection of R with the interior of K_n is the interior of T_n, the boundary of R is a rectifiable simple closed curve, and the closure of R (with the exception of the boundary point t) lies in D.

Let F be a function which is one-to-one and continuous on $|z| \leq 1$, analytic in $|z| < 1$, and which maps $|z| \leq 1$ onto the closure of R, such that $F(1) = t$. Since the boundary of R is rectifiable, Theorem 1(ii) shows that f is in A^+, if $f(\theta) = F(e^{i\theta})$.

On the other hand, the image C of the interval $[0, 1]$ under the mapping F is an arc which traverses every cell T_n from a point on A_n to a point on B_n. Recalling that $|\phi'(w)| > n$ in T_n and that the length of $C \cap T_n$ is at least n^{-2}, we see that the image of C under ϕ (which is the same as the image of $[0, 1]$ under $\phi \circ F$) is a non-rectifiable curve. By Theorem 1(i), $\phi \circ f$ does not belong to A, and the theorem is proved.

THEOREM 3. *There exists a function ψ in A such that $\psi(\theta) > 0$ if $\theta \neq 0$, $\psi(0) = 0$, and $\sqrt{\psi}$ is not in A.*

Proof: Applying Theorem 2 with the right half plane for D, $\phi(w) = \sqrt{w}$, and $t = 0$, we see that there exists a function $q = u + iv$, where u and v are real and v is the conjugate of u, with the following properties: $q(0) = 0$, $u(\theta) > 0$ if $\theta \neq 0$, q is not in A, but q^2 is in A^+. Since the Fourier coefficients of q with negative index are zero, u is not in A either. We put $f = q^2$.

If $|f|$ is not in A, take $\psi = |f|^2$, which is in A, since $|f|^2 = f \cdot \bar{f}$. Suppose that $|f|$ is in A. Since f is in A, so is its real part $u^2 - v^2$. Also, $|f| = u^2 + v^2$. Adding, we see that u^2 is in A, and $\psi = u^2$ satisfies the conditions of the theorem.[7]

Replacing θ by x, we now consider the function ψ as a periodic function on the real line; define a function k by: $k(x) = 1$ if $|x| < \pi/2$, $k(x) = 0$ if $|x| > 3\pi/4$, $k(x)$ linear in the remaining intervals; let $h = k \cdot \psi$; and define g by: $g^2(x) = h(x)$, $g(x) \geq 0$.

Using the well-known relation between absolutely convergent Fourier series and absolutely convergent Fourier integrals, together with Wiener's localization theorem,[8] we see that h is the Fourier transform of a function H in L_1, but g is not the Fourier transform of any such function. Also, H is an entire function (even of exponential type), since h vanishes outside a bounded interval; and H is positive-definite,[9] since h is nonnegative, so that we may state the following theorem:

THEOREM 4. *There exists an entire function H, of exponential type, which belongs to L_1 on the real line and is positive-definite, such that the convolution equation*

$$H(x) = \int_{-\infty}^{\infty} G(x - t)G(t)\, dt \qquad (x \text{ real})$$

has no positive definite solution G in L_1.

We conclude with the remark that g is not even the Fourier-Stieltjes transform of any bounded (complex-valued) Borel measure on the line. For, if it were, let $k_1(x) = k(x/2)$; then k_1 has its Fourier transform in L_1. Since the convolution of a member of L_1 and a bounded measure is in L_1, it follows that $g \cdot k_1$ has its transform in L_1. But $g \cdot k_1 = g$, and we arrive at a contradiction.

[1] N. Wiener, "Tauberian Theorems," *Ann. Math.*, **33**, 1–100, 1932.

[2] P. Lévy, "Sur la convergence absolue des séries de Fourier," *Compositio Math.*, **1**, 1–14, 1934.

[3] The range of a function is the set of its values.

[4] $\phi \circ f$ is the function defined by: $(\phi \circ f)(\theta) = \phi(f(\theta))$.

[5] For a simple proof see A. Zygmund, *Trigonometrical Series*, Warsaw: Monografje Matematyczne (1935), p. 158.

[6] A topological 2-cell is a topological image of a closed disk.

[7] Our proof of Theorem 3 is based on Theorem 2. Quite recently (*Compt. Rend. Acad. Sci. Paris*, January 3, 1955) J. P. Kahane has constructed, by fairly elaborate computations, an example of a real function f in A such that $|f|$ is not in A; this also implies Theorem 3.

[8] Wiener, *op. cit.*, pp. 10, 14 (Lemmas II*b*, II*e*).

[9] See, for instance, S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig: Akademische Verlagsgesellschaft (1932), p. 74.

ON THE GENERATING PARAMETRIX OF THE STOCHASTIC PROCESSES

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1. Let X be a locally compact space, and let $\mathfrak{B}$ be the Borel field generated from the open sets of X. A temporally homogeneous discrete Markoff process (d.M.p.) in X is defined by the transition probability $G(x, E)$ satisfying the conditions

$$G(x, E) \geqq 0,\ G(x, X) = 1, \qquad (1.1)$$

$$G(x, E) \text{ is countably additive in } E \in \mathfrak{B} \text{ and } \mathfrak{B}\text{-measurable in } x. \qquad (1.2)$$

A temporally homogeneous Markoff process (M.p.) in X is defined by a one-parameter family of transition probabilities $P(t, x, E)$, $t > 0$, satisfying the conditions

$$P(t, x, E) \geqq 0, P(t, x, X) = 1, \tag{1.1}'$$

$$P(t, x, E) \text{ is countably additive in } E \in \mathfrak{B} \text{ and } \mathfrak{B}\text{-measurable in } x, \tag{1.2}'$$

$$P(t + s, x, E) = \int_X P(t, x, dy)P(s, y, E). \tag{1.3}$$

We assume[1] the "continuity condition"

$$\lim_{t \downarrow 0} P(t, x, X - V(x)) = 0 \text{ for any vicinity } V(x) \text{ of } x. \tag{1.4}$$

Hence we see, by (1.3), that $P(t, x, E)$ is measurable in t. The integral

$$G(x, E) = \int_0^\infty \exp(-t)P(t, x, E)\, dt \tag{1.5}$$

defines a d.M.p.

The purpose of the present note is to show that (see, for the exact formulation, the theorem in sec. 2) a certain class $\mathfrak{P}$ of $P(t, x, E)$ is related in one-one manner with a certain class $\mathfrak{G}$ of $G(x, E)$ in such a way that

$$\mathfrak{G} \ni G(x, E) = \int_0^\infty \exp(-t)P(t, x, E)\, dt \quad \text{if } P(t, x, E) \text{ is } \in \mathfrak{P}, \tag{1.6}$$

$$\mathfrak{P} \ni P(t, x, E) = \lim_{\alpha \downarrow 0} \exp\left(t(G(x, E) - \delta(x, E))/(\alpha\delta(x, E) + (1 - \alpha)G(x, E))\right) \text{ if } G(x, E) \text{ is } \in \mathfrak{G} \text{ and if } \delta(x, E) \text{ denotes the Dirac measure:} \tag{1.7}$$

$$\delta(x, E) = 1 \text{ or } 0 \text{ according as } x \in E \text{ or not.} \tag{1.8}$$

We shall call $G(x, E)$ of the class $\mathfrak{G}$ as "the generating parametrix" of the M.p. $P(t, x, E)$. The construction and the properties of the M. processes $\in \mathfrak{P}$ are thus reduced to those of the d.M. processes $\in \mathfrak{G}$. As an application, it will be shown in section 3 that we may decompose X (under a M.p. $P(t, x, E)$) into ergodic parts and dissipative part.

2. Let $C(X)$ be a Banach space consisting of a certain linear set of bounded continuous real-valued functions $f(x)$ with the norm

$$\|f\| = \sup_x |f(x)|, \tag{2.1}$$

such that $C(X)$ contains all the continuous functions whose carriers are compact and are contained in the interior of X.

Let $\mathfrak{P}$ be the class of M. processes $P(t, x, E)$ in X satisfying (1.1)', (1.4) and the condition

$$(P_t f)(x) = \int_X P(t, x, dy)f(y) \text{ defines a linear operator on } C(X) \text{ into } C(X). \tag{2.2}$$

Let $\mathfrak{G}$ be the class of d.M. processes $G(x, E)$ in X satisfying (1.1), (1.2) and the conditions

$$(Gf)(x) = \int_X G(x, dy)f(y) \text{ defines a one-one mapping of } C(X) \text{ onto a strongly dense linear subspace of } C(X); \tag{2.3}$$

for any α, $0 < \alpha < 1$, the linear operator $\alpha I + (1 - \alpha)G$ (I = the identity operator) defines a one-one mapping of $C(X)$ onto $C(X)$ such that the linear operator $G_\alpha = G(\alpha I + (1 - \alpha)G)^{-1}$ on $C(X)$ into $C(X)$ is defined by a d.M.p. $G_\alpha(x, E)$ through $(G_\alpha f)(x) = \int_X G_\alpha(x, dy)f(y)$. (2.4)

Then we have the

THEOREM. For any $P(t, x, E) \in \mathfrak{P}$, $G(x, E) = \int_0^\infty \exp(-t)P(t, x, E)\,dt$ belongs to $\mathfrak{G}$. Conversely, any $G(x, E) \in \mathfrak{G}$ defines $P(t, x, E) \in \mathfrak{P}$ by the linear operator

$$P_t f = \text{strong} \lim_{\alpha \downarrow 0} \exp(t(G - I)(\alpha I + (1 - \alpha)G)^{-1}) \text{ (uniformly in any bounded interval of } t) \tag{2.5}$$

in such a way that

$$(P_t f)(x) = \int_X P(t, x, dy)f(y) \text{ and } G(x, E) = \int_0^\infty \exp(-t)P(t, x, E)\,dt.$$

The proof may be reduced to the theory of semigroup of linear operators[2] through the

LEMMA. Let $\{P_t\}$, $t > 0$, be a semigroup

$$P_t P_s = P_{t+s} \tag{2.6}$$

of bounded linear operators on a Banach space S into S such that

$$\text{weak} \lim_{t \downarrow 0} P_t f = f \text{ for every } f \in S. \tag{2.7}$$

Then we have

$$\text{strong} \lim_{t \to 0} P_t f = f \text{ for every } f \in S. \tag{2.8}$$

Proof: For any $f_0 \in S$, the strong closure $\bar{S}_0$ of the linear subspace S_0 spanned by $\{P_t f_0;\ t > 0\}$ is mapped into $\bar{S}_0$ by every P_t. This we see by (2.6). By the equivalence of the strong closure with the weak closure in the Banach space, we see, by (2.6)–(2.7), that $\bar{S}_0$ is strongly separable and that $f_0 \in \bar{S}_0$. P_t is weakly measurable on $\bar{S}_0$ by (2.7). Also, by (2.7), $\|P_t\|$ is bounded in t. Hence, by a theorem of N. Dunford,[3] we see that P_t is, for $t > 0$, strongly continuous in t on $\bar{S}_0$. Thus we see, by (2.6)–(2.7), that $\text{strong} \lim_{t \to 0} P_t f_0 = f_0$.

Proof of the Theorem: Let $P(t, x, E)$ be $\in \mathfrak{P}$, and let $f \in C(X)$. For any $x \in X$ and for any $\epsilon > 0$, let the vicinity $V(x_0)$ of x_0 be chosen such that $x \in V(x_0)$ implies $|f(x) - f(x_0)| < \epsilon$. Then we have

$$(P_t f)(x_0) - f(x_0) = \int_{V(x_0)} P(t, x_0, dx)(f(x) - f(x_0)) + \int_{x - V(x_0)} P(t, x_0, dx)(f(x) - f(x_0)).$$

Thus, by (1.4), we see that

$$\lim_{t \to 0} (P_t f)(x_0) = f(x_0) \text{ boundedly in } x_0. \tag{2.9}$$

Hence $\text{weak} \lim_{t \downarrow 0} P_t f = f$. We have, by (1.3), (2.6) also. Thus we have, by the lemma,

$$\text{strong} \lim_{t \downarrow 0} P_t f = f \text{ for every } f \in C(X). \tag{2.10}$$

Therefore, we may apply the semigroup theory to $\{P_t\}$. Let A be the infinitesimal generator of P_t, viz., let

$$Af = \text{strong} \lim_{t \downarrow 0} t^{-1}(P_t - I)f \text{ if the right-hand limit exists.} \tag{2.11}$$

Such A is characterized by the following properties:

the domain $D(A)$ of A is strongly dense in $C(X)$, (2.12)

for any $\alpha > 0$, the resolvent $G_\alpha = (I - \alpha A)^{-1}$ exists as a bounded linear operator on $C(X)$ into $C(X)$ such that (2.13)

$G_\alpha 1 = 1$ and $(G_\alpha f)(x)$ is nonnegative for nonnegative function $f(x)$. (2.14)

Moreover, we have

$$G_\alpha = \alpha^{-1} \int_0^\infty \exp(-\alpha^{-1}t) P_t \, dt. \tag{2.15}$$

Thus we have

$$AG_\alpha = \alpha^{-1}(G_\alpha - I) \tag{2.16}$$

and hence

$$(I - \alpha A)G_1 = G_1 - \alpha(G_1 - I) = \alpha I + (1 - \alpha)G_1 \tag{2.17}$$

gives a one-one linear mapping of $C(X)$ onto $C(X)$. Hence

$$(\alpha I + (1 - \alpha)G_1)^{-1} = (I - A)G_\alpha,$$

and thus

$$G_\alpha = G(\alpha I + (1 - \alpha)G)^{-1}, \text{ where } G = G_1. \tag{2.18}$$

We have thus proved the first half of the theorem. The latter half of the theorem is proved also by the semigroup theory. In fact, the conditions (2.3)–(2.4) imply the existence of the uniquely determined semigroup

$$\begin{aligned} P_t f &= \text{strong} \lim_{\alpha \downarrow 0} \exp(tAG_\alpha) = \text{strong} \lim_{\alpha \downarrow 0} \exp(tAG(\alpha I + (1 - \alpha)G)^{-1}) \\ &= \text{strong} \lim_{\alpha \to 0} \exp(t(G - I)(\alpha I + (1 - \alpha)G)^{-1} \end{aligned} \tag{2.19}$$

strongly continuous in t with the infinitesimal generator

$$A = I - G^{-1}. \tag{2.21}$$

That this P_t is defined by a M.p. may be seen from

$$\begin{aligned} \exp(t(G - I)(\alpha I + (1 - \alpha)G)^{-1}) &= \exp(tAG_\alpha) \\ &= \exp(t\alpha^{-1}(G_\alpha - I)) \\ &= \exp(-\alpha^{-1}t) \exp(t\alpha^{-1}G_\alpha) \end{aligned} \tag{2.22}$$

and (2.14). The continuity condition (1.4) of the corresponding $P(t, x, E)$ may be proved by the strong continuity (2.11) of P_t.

Remark: We have, by (2.16),

$$AG = G - I, \text{ where } G = G_1. \tag{2.16}'$$

Thus $G(x, E)$ is the "parametrix" of the infinitesimal generator A of P_t. Hence we call $G(x, E)$ as the "generating parametrix" of the Markoff process $P(t, x, E)$.

3. A nonnegative, normalized measure $\varphi(E)$, countably additive for $E \in \mathfrak{B}$, is called an "invariant measure" of the process $P(t, x, E)$ if

$$\varphi(E) = \int_X \varphi(dx)P(t, x, E) \text{ for every } t > 0 \text{ and for every } E \in \mathfrak{B}. \tag{3.1}$$

It is easy to see, by the theorem above in section 2, that $\varphi(E)$ is invariant for $P(t, x, E)$ if and only if $\varphi(E)$ is invariant for the corresponding $G(x, E)$:

$$\varphi(E) = \int_X \varphi(dx)G(x, E) \text{ for every } E \in \mathfrak{B}. \tag{3.2}$$

Let us consider the special case when X is separable and when the operator G maps continuous functions with compact carriers into continuous functions with compact carriers. In another paper[4] the present author proved the following extension of the results due to J. von Neumann,[5] N. Kryloff and N. Bogoliouboff,[6] and M. Beboutoff:[7]

X is decomposable into a dissipative part D and ergodic parts E_λ in such a way that

$$\varphi(D) = 0 \text{ for every invariant measure } \varphi, \tag{3.3}$$

$$G(x, E_\lambda) = 1 \text{ for every } x \in E_\lambda, \tag{3.4}$$

$G(x, E)$, considered (by (3.4)) as a d.M.p. in E_λ, has one and only one invariant measure φ_λ. (3.5)

It is easy to see, by the theorem in section 2, that the above decomposition of X with respect to the d.M.p. $G(x, E)$ gives also the decomposition of X into a dissipative part and ergodic parts with respect to the corresponding M.p. $P(t, x, E)$ for every t simultaneously.

[1] Recently, W. Feller has published interesting results concerning the general form of the infinitesimal generator of $P(t, x, E)$ for the case, on real line X, where the "Lindeberg's condition" $\lim_{t \downarrow 0} t^{-1}P(t, x, X - V(x)) = 0$ is satisfied. See, for example, his paper "The General Diffusion Operator and Positivity Preserving Semi-groups in One Dimension," *Ann. Math.*, **60**, No. 3, 417–436, 1954. In this connection see also K. Yosida, "On Brownian Motion in a Homogeneous Riemannian Space," *Pacific J. Math.*, **2**, 263–270, 1952.

[2] E. Hille, *Functional Analysis and Semi-groups*, (New York, 1948), and K. Yosida, "On the Differentiability and the Representation of One Parameter Semi-group of Linear Operators," *J. Math. Soc. Japan*, **1**, 15–21, 1948.

[3] "On One-Parameter Groups of Linear Transformations," *Ann. Math.*, **39**, 569–573, 1938.

[4] "Simple Markoff Process with a Locally Compact Phase Space," *Mathematica Japonica*, **1**, No. 3, 1–5, 1948.

[5] "Zur Operatorenmethode in der klassischen Mechanik," *Ann. Math.*, **33**, 587–642, 1932.

[6] "La Théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire," *Ann. Math.*, **38**, 65–113, 1947.

[7] "Simple Markoff Process with a Compact State Space," *Rec. Math.*, **10**, 22–30, 1942.

AN APPLICATION OF THE POISSON-BOLTZMANN EQUATION TO NONIDENTICAL, CHARGED PARTICLES

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The theory of the interaction of large, high-charge bearing, identical particles immersed in an electrolytic medium has been placed on a firm footing by Verwey and Overbeek.[1] If an attempt is made to generalize this theory to nonidentical particles, its conclusions become physically unacceptable and a revision of some of its basic assumptions is necessitated. In the discussion which follows, an acquaintance with the Verwey-Overbeek theory is assumed.

1. *Application of the Verwey-Overbeek Theory to Two Nonidentical Particles.*— Consider two infinite parallel plates (the first at $x = 0$, the second at $x = l$) in an ionic medium. The electric potential between them is found from the solution of the Poisson-Boltzmann (P.B.) equation, as developed by Debye and Hückel. Verwey and Overbeek[2] impose the boundary conditions that the surface potentials, $\psi(0)$ and $\psi(l)$, are independent of the interaction, i.e., the separation distance l. A family of curves results from the above, giving $\psi(x)$ as a function of l; these are shown in Figure 1 for two identical particles, $\psi(0) = \psi(l)$. It is seen that the minimum of $\psi(x)$ increases with increasing interaction and that the curve approaches a horizontal line in the limit as l approaches zero. An examination of the equation shows that the derivative with respect to l, and hence the charge densities of the two plates (from Gauss's law), approach zero for large interaction.

But, if we now fix $\psi(0) \neq \psi(l)$ and cause these two *nonidentical* plates to approach, $\psi(x)$ must clearly acquire a larger and larger slope which will ultimately become infinite. The charge densities therefore also become unbounded, a conclusion which is physically unacceptable. The above physical argument (represented in Fig. 2) can be confirmed by an examination of the first and second integrations of the P.B. equation. Further, if we solve the linear approximation of the P.B. equation, we find that the force between these plates becomes infinite and attractive at $l = 0$.

These results point up the need for a revision of the theory. In particular, they indicate the need for a revision of the condition that the surface potentials remain independent of the particle interaction. This boundary condition is justified in the Verwey-Overbeek theory by the following argument. The charge on the surface has its origin in the adsorption of potential-determining ions from the solution. There must exist, therefore, a drop in the chemical potential $\Delta\mu$ between these two phases. The adsorption is opposed by the electric potential difference, $\psi(0)$. At equilibrium these two must balance, i.e.,

$$\epsilon\nu\psi(0) + \Delta\mu = 0, \tag{1}$$

where ϵ is the elementary electric charge and ν the valency. The chemical potential $\Delta\mu$ depends on two factors, the ion concentration c in solution and that on the

surface. If the surface concentration is ignored, as is done by Verwey and Overbeek, $\psi(0)$ becomes

$$\psi(0) = \frac{kT}{\epsilon\nu} \ln \frac{c}{c_0}, \tag{2}$$

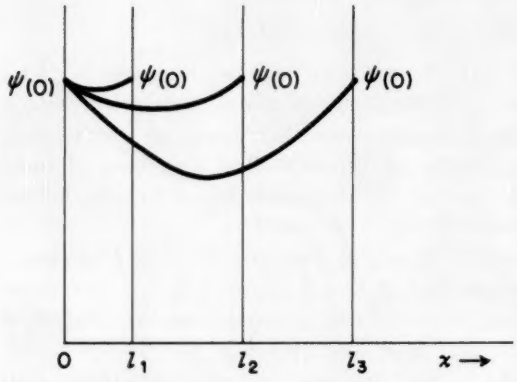

FIG. 1.—The family of curves $\psi(x)$ for various l, for two identical particles.

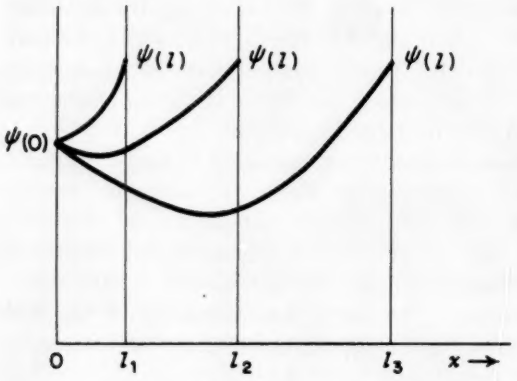

FIG. 2.—The family of curves $\psi(x)$ for various l, for two nonidentical particles.

where c_0 is the ion concentration in solution when the particle is at its isoelectric point. From equation (2), $\psi(0)$ is independent of the particle interaction.

Since we desire to abandon the independence of $\psi(0)$ and $\psi(l)$ of l, we must include the surface concentration of the potential-determining ion in calculating the chemical potential change. This inclusion leads directly to the revised theory.

2. *The Revised Theory.*—Consider a surface of an infinite plate immersed in an electrolytic medium. A given kind of ion (say, a cation) can migrate from the solution and is adsorbed on open sites of the surface. The plate has also a given negative charge density, which is assumed here to be independent of the ions in solution. Let N^* be the total number of these sites available per unit area to the cation; let N_0 be the number of these sites filled when the surface is neutral; and let c_0 be the ion concentration at infinity in solution when the plate is isoelectric. Each plate is fully characterized by these three independent parameters. Further, let N and c represent the surface and medium concentrations of the ion, respectively, for some arbitrary state of the system. If we now assume that the cations do not interact in any way on the surface except for the crowding effect, their chemical potentials on the surface and in solution are given by[3]

$$\mu \text{ (surface)} = \mu_{0s}(T, P) + kT \ln \frac{N}{N^* - N},$$

$$\mu \text{ (solution)} = \mu_{0m}(T, P) + kT \ln c. \tag{3}$$

If N sites are filled, and the ion concentration in solution is c, $\Delta\mu$ is given by

$$\Delta\mu = \mu_{0s} - \mu_{0m} + kT \ln \frac{N}{c(N^* - N)},$$

or, considering the definition of c_0 and N_0,

$$\Delta\mu = kT \ln \frac{c_0 N(N^* - N_0)}{cN_0(N^* - N)}. \tag{4}$$

At equilibrium, therefore,

$$\psi(0) = \psi_0 = \frac{kT}{\epsilon\nu}\left[\ln \frac{c}{c_0} - \ln \frac{N(N^* - N_0)}{N_0(N^* - N)}\right]. \tag{5}$$

In calculating the free energy of this system, we follow the method proposed by Verwey and Oberbeek, i.e., slowly charging the surface by bringing up charges from infinity and considering the free energy changes due to (1) the chemical potential drop and (2) the work done against the electric potential. The chemical contribution ΔF_c is

$$\Delta F_c = \int_{N_0}^{N} \Delta\mu \, dN' = kT\left\{(N - N_0) \ln \frac{c}{c_0} - N \ln \frac{N}{N_0} + (N^* - N) \ln \frac{N^* - N_0}{N^* - N}\right\}, \tag{6}$$

which differs from the Verwey-Overbeek theory by the last two terms. The electric contribution is given by $\int_0^\sigma \psi(\sigma')\, d\sigma'$.

3. *Application of the Revised Theory to Two Parallel Plates.*—In evaluating the revised theory we shall limit ourselves in this paper to a consideration of the linear form of the P.B. equation, which restricts the validity of the theory to small potentials.[4]

The linear P.B. equation $\nabla^2\psi = K^2\psi$, when evaluated for two parallel plates, $\psi(0) = \psi_{01}$, $\psi(l) = \psi_{02}$, results in the following:

$$\sigma_i = \epsilon\nu(N_i - N_{0i}) = \frac{DK}{4\pi \sinh Kl}[\psi_{0i} \cosh Kl - \psi_{0j}].$$

$$\psi_{0i} = \frac{4\pi}{DK \sinh Kl}[\sigma_i \cosh Kl + \sigma_j], \qquad i \neq j, \tag{7}$$

where σ_i is the charge density on the ith plate and D the dielectric constant of the medium. From equations (7) and (5) we can solve for $N_i - N_{0i}$ when the plate is at equilibrium:

$$\gamma \sinh Kl\,(N_i - N_{0i}) = Q_i(l) \cosh Kl - Q_j(l), \tag{8}$$

where $\gamma = 4\pi\epsilon^2\nu^2/DkT$ and $Q_i(l) = \ln [cN_{0i}(N_i^* - N_i)/c_{0i}N_i(N_i^* - N_{0i})]$.

Differentiation of equation (8) yields

$$\frac{dN_i}{dl} = \frac{K\gamma[G_j(N_j - N_{0j}) + F_j(N_i - N_{0i})]}{G_iG_j - F_iF_j}, \tag{9}$$

where

$$G_i = \gamma \sinh Kl + \frac{N_i^* \cosh Kl}{N_i(N_i^* - N_i)}, \tag{10}$$

$$F_i = \frac{N_i^*}{N_i(N_i^* - N_i)}.$$

We can now evaluate $N_i - N_{0i}$ by expanding equation (8) around the origin. It can be shown very simply that the charge density is *only finite at* $l = 0$ *if* $Q_i(0) = Q_j(0)$, a result in complete agreement with our physical argument at the beginning of the paper as indicated in Figures 1 and 2. Furthermore, it follows from the expansion that $\sigma_i(0) = -\sigma_j(0)$, i.e., the charge density becomes reversed on one plate and equals in magnitude the positive charge density on the other.

To calculate the free energy, we proceed in the same general manner as Verwey and Overbeek, with suitable modifications for a nonidentical system.

Assume that at large distance plate 1 has a smaller positive charge than plate 2. Starting off with both plates electrically neutral, they are gradually charged up in an identical fashion until both simultaneously reach that charge density $\bar{\sigma} = \epsilon\nu(\bar{N}_i - N_{0i})$ which plate 1 would have at equilibrium if it were interacting with another plate identical to itself. We then turn to plate 2 and charge it up to the equilibrium value desired for this system. Plate 1, during this second process, is always in thermodynamic equilibrium and will by itself reach its proper charge density without contributing to the ΔF.

In following this indicated procedure, the calculations are completely straightforward with one consideration. In the second charging process, when one calculates the electrical free energy change, one must evaluate the integral $\int_{\bar{\sigma}}^{\sigma_2} (\sigma_2 \cosh Kl + \sigma_1)\, d\sigma_2$. This necessitates finding dN_2/dN_1, which follows directly from equation (5) and (7). Omitting all the calculations, we put down the result of the above:

$$\frac{\Delta F}{KT} = -(N_2 - N_{02}) \ln \frac{cN_{02}}{c_{02}(N_2^* - N_{02})} - N_{01} \ln \frac{N_{01}}{N_1^* N_{01}} + N_1^* \ln \frac{N_1^* - N_1}{N_1^* - N_{01}} -$$
$$N_{02} \ln \frac{N_{02}}{N_2^* - N_{02}} + N_2^* \ln \frac{N_2^* - N_2}{N_2^* - N_{02}} - N_2 \ln \frac{N_2^* - N_2}{N_2} - N_{01} \ln \frac{N_1^* - N_1}{N_1} +$$
$$\frac{\gamma \cosh Kl}{2 \sin h\, Kl} [(N_2 - N_{02})^2 - (N_1 - N_{01})^2]. \tag{11}$$

The force between the plates is found from a differentiation of equation (11) with respect to l. Again we only state the result:

$$\frac{P}{kT} = \frac{1}{kT} \frac{\partial \Delta F}{\partial l} = \frac{-\gamma}{2 \sinh^2 Kl} \left\{ K[(N_2 - N_{02})^2 - (N_1 - N_{01})^2] + \right.$$
$$\left. 2 \sinh Kl\, (N_1 - N_{01}) \left(\frac{dN_2}{dl} + \cosh Kl \frac{dN_1}{dl} \right) \right\} - \frac{dN_1}{dl} \frac{N_1^*(N_1 - N_{01})}{N_1(N_1^* - N_1)}. \tag{12}$$

When equation (12) is evaluated for two identical plates, i.e., $1 = 2$, the expression, after suitable manipulations, reduces to:

$$\frac{P}{KT} = \frac{K\gamma(N - N_0)^2}{\sinh^2 Kl} (\cosh Kl + 1), \tag{13}$$

which is formally identical with the Verwey-Overbeek expression.

It is also apparent from equation (12) that the force between two nonidentical plates is always repulsive as long as the charge densities are both positive. But, since the charge density of one plate reverses its sign for large interaction, equation (12) must be examined more carefully.

The force is evaluated for small l by expanding equation (12) around the origin, using equations (7), (8), (9), and (10). The calculations are rather lengthy, and we shall simply state the result. It is found that, for $l = 0$, the force is finite and equals

$$\frac{P}{kT} = \frac{K}{2\gamma}\left[\frac{\gamma^2\sigma_1(0)^2}{\epsilon^2\nu^2} - Q(0)^2\right]. \tag{14}$$

This expression reduces to $-(K/2\gamma)Q(0)^2$ for two identical plates, in agreement with equation (13), and is of course repulsive. Clearly, then, the force between *two nonidentical* plates is always *less repulsive* for $l = 0$ than between two *identical* plates and can become attractive for $\gamma|\sigma_1(0)| > |\epsilon\nu Q(0)|$. These two expressions can be evaluated from $Q_1(0) = Q_2(0)$, and $\sigma_1(0) = -\sigma_2(0)$, and the above inequality can be clearly satisfied for certain values of the parameters of the two plates. Since $Q(0)$ is proportional to $\psi(0)$, it is apparent that attractive forces will arise for small potentials, which in turn justifies our use of the linear form of the P.B. equation.

We find, then, that the revised theory obviates the difficulties of the Verwey-Overbeek theory with respect to nonidentical plates. Everything remains finite at all distances, and, if a certain inequality holds, the theory predicts that short-range attractive forces arise between these particles. We explain these attractive forces by pointing out that two positively charged particles, if made to approach, drive charges off each other's surfaces. If they are identical, the charge density simply goes to zero in a symmetrical manner. If they are nonidentical, one surface will become neutral and then negative when the other is still positive. In this region two forces exist: an electrostatic attractive force between two oppositely charged particles and a repulsive force due to the affinity of the cations in solution for the plate surfaces. These ions attempt to migrate back to their sites (with an affinity which increases with decreasing charge density) and hence exert a counteracting influence, causing the plates to move apart. Under certain specified conditions the net force becomes attractive in this region. The difficulty with the Verwey-Overbeek theory is now also apparent. In that theory the affinity does not increase with loss of charge; hence a charge reversal is not counteracted, leading to an infinite accumulation of negative charge.

We find, therefore, in the revised theory that it is no longer necessary to resort to London–van der Waals forces in order to account for short-range attraction between nonidentical particles. This, of course, in no way rules out the probable importance of these forces.

It is also of interest that the direction as well as the magnitude of the force between large particles is very much determined by their identity or nonidentity.

Specifically, two particles always repel; two nonidentical particles may attract; they definitely repel less. This is clearly of some significance in biology, where the interactions of particles are very much determined by whether they are identical or not. In mitosis, for example, we must postulate that a chromosome is somehow capable of attracting to itself various nonidentical smaller molecules and synthesizing them into a replica of itself; then, when an identical particle has been produced, a repulsive force must arise, causing them to separate. It is clear that the required relationships between force and identity in the above situation are the same as in the revised theory.

The author hopes to publish in the near future a more extensive paper, giving more of the calculations upon which the above results are based, as well as additional consequences of the theory.

4. *Conclusions.*—Since the Verwey-Overbeek theory cannot be generalized to nonidentical particles, a revised theory is developed. When applied to identical particles, it gives the same expression for the force as the Verwey-Overbeek theory. It leads also to an acceptable expression for the force and charge density between two nonidentical particles and predicts the existence of short-range attractive forces (under specified conditions) between particles both positively charged at large separation. A physical explanation is offered for the origin of this attractive force.

The author wishes to express his appreciation to his wife, Enid H. Bierman, for her scientific and moral assistance.

[1] E. J. W. Verwey and J. Th. G. Overbeek, *Theory of the Stability of Lyophobic Colloids* (New York: Elsevier Publishing Co., Inc., 1948).

[2] *Ibid.*, p. 61.

[3] R. H. Fowler and E. A. Guggenheim, *Statistical Thermodynamics* (Cambridge: The University Press, 1939).

[4] The author has recently evaluated the complete nonlinear form of the P.B. equation with respect to nonidentical plates. The results bear out completely the conclusions of this paper, and will be published in the near future.

ERRATA: ON GENERATING FUNCTIONS FOR RESTRICTED PARTITIONS OF RATIONAL INTEGERS

In the article of the above title appearing in these PROCEEDINGS, **41**, 37–42, 1955, the following corrections should be made:

Equation (8) should read:

$$xF_n'(x) = x(1 - x^n)F_{n-1}'(x) - nx^nF_{n-1}(x).$$

Equation (18) should read:

$$\Phi(r, n) = \sum_{p=0}^{\eta r} A_{pn+r}^{(n-1)}.$$

The line below equation (18a) should read:

"where $D_r = \sum_{b=0}^{\mathfrak{m}_r} A_{bl+r}^{(n)}$ and $\mathfrak{m}_r = [(n - r)/l]\ldots$."

C. A. NICOL and H. S. VANDIVER

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